WKB ESTIMATES TO THE CRITICAL LENGTH OF TWISTED SOLAR CORONAL LOOPS

Simon I. Herbert

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Abstract

The solar corona exhibits many different phenomena, observable from the Earth or space. Magnetohydrodynamic stability theory provides a method of investigating these phenomena by using it to test proposed mathematical models. WKB is a way of approximating the solutions of second order linear homogeneous differential equations with large parameters and so together with MHD stability theory, models for solar coronal loops can be investigated. In this thesis, the problem of a line tied twisted coronal loop is studied within the framework of ideal MHD using a WKB approximation to estimate the critical length at which the various magnetic fields become unstable.

The problem will be split into two halves: (i) force-free and (ii) non force-free fields. Using a finite element/Fourier method, the full MHD equations will be solved numerically and the results compared with analytical solutions.

Declarations

I, Simon Ian Herbert, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

Signed Date......2/6/95..........

I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No. 12 in October 1993 and re-registered for the degree of MPhil in January 1995.

Signed Date......2/6/95..........

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of MPhil.

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Acknowledgements

Leaving the hustle and bustle of London University to come to what my friends considered to be the equivalent of darkest Peru, to study the sun where I was told it constantly rained was strange. Spending two years looking at equations about something you hardly ever saw... quick, ring the funny farm! A huge thanks goes to Alan, my supervisor, for guiding me down the seemingly unending path of research and for steering me round the odd brick wall that cropped up. To Ronald Van Der Linden for all his help with the numerical side. Thanks also to Gordon, Tim and Luca, the various inmates of room 223 (sorry about my untidyness), the Solar Theory Group as a whole (but whether it's Cheryl or Jason that gets my desk I can't say) and the EPSRC for providing me with financial support. Another huge thanks goes to my parents whom I've never quite thanked properly for all their moral and financial support throughout my entire education. The biggest thank you of all however, must go to my girlfriend Paula for putting up with all my moaning for the past year and for being there when I needed her most. Thank you everyone.

To P, the one true star in my life.

'I found it hard,
it was hard to find,
oh well, whatever,
nevermind.'

(Nirvana, 'Smells like teen spirit', 1991)

Contents

1 Introduction

In the heavens he has pitched a tent for the sun,
which is like a bridegroom coming forth from his pavilion,
like a champion rejoicing to run his course.
It rises at one end of the heavens and makes its circuit to the other;
nothing is hidden from its heat.
(Psalm 19:5)

To the average person on Earth the Sun in all its splendour is nothing more than an object in the sky providing us with heat and light; is it really important to our understandings of future life? Apart from the obvious, 'well where would life be without it?', it is a vat of information spilling over with new ideas for scientists on Earth for the development of research into other galactic phenomena, such as variable stars and nebulae. Why our Sun and not another more interesting star; purely because it is in our own backgarden, just ripe for the picking and who needs a ninety three million mile long ladder when modern technology provides us with hard/soft X-ray pictures, magnetograms and coronagraphs.

Observations are the backbone in any research environment, they provide hard evidence for theory to evolve and prove itself (Galileo on the leaning tower of Pisa is testament to this). It was not until hard observations of the sun were made that solar theory really took off, whence correlations were made between observed phenomena on the sun and other stars in the universe. The solar atmosphere has been observed to be constructed from three regions: the photosphere (from which most visible light is emitted and on whose surface sunspots have been observed since the time of Aristotle), the chromosphere (so called after the Greek word chromo meaning colour, which is studied to understand the evolution of active regions and prominences) and the corona (the very outer atmosphere of the sun, visible to the naked eye as a white halo only at total eclipse and responsible for the subject of this thesis: coronal loops).

Coronal loops are magnetic loops of plasma typically 5×10^5km long with a temperature between 2 to 3×10^6K and a number density of $7 \times 10^{14}\mathrm{m}^{-3}$. The footpoints of the magnetic field lines are rooted in the dense photospheric plasma and the loops themselves last for about a day, often erupting to form compact flares. It is the length at which these loops become unstable that is the subject of the following work: their critical length.

Diversifying, when completing a jigsaw puzzle the easiest way to do it is to complete the outer edge first, because those pieces are the easiest to find (they all have straight edges), then the rest follows (apart from the annoying last piece that the dog ate). The analogy is saying that the outer edge holds everything in place, it is the theory behind the practice. Solar theory has this element, magnetohydrodynamics (MHD for short) is the 'outer edge' that holds all solar research together and therefore is the basic model on which the work in this thesis is based.

1.1 Basic Equations of MHD

The essential feature of MHD is the interaction of magnetic fields with the motion of conducting fluids. The basic principles were understood in the 1830's, but MHD theory did not really develop until this century when it was discovered that the sun, planets and stars all have magnetic fields. Dissipative effects are neglected creating the ideal MHD state, the governing equations for which are

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{1.2}$$

$$\frac{\partial p}{\partial t} = -(\mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v}), \tag{1.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{1.4}$$

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}, \tag{1.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.6}$$

$$p = \frac{\mathcal{R}}{\tilde{\mu}} \rho T, \tag{1.7}$$

where $\mathbf{B}$ is the magnetic induction, normally referred to as the magnetic field, p is the gas pressure, ρ is the mass density, $\mathbf{j}$ is the electric current, t is time and T is temperature. The constants γ, μ, $\mathcal{R}$ and $\tilde{\mu}$ are the adiabatic index, the magnetic permeability, the ideal gas constant and the mean molecular weight respectively. Equation (1.1) is the momentum equation where the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

gives the Lagrangian (following the motion) time rate of change in terms of Eulerian (fixed point) measurements. Equation (1.2) is the mass continuity equation, (1.3) is the adiabatic energy equation, (1.4) is the induction equation, (1.5) is Ampere's law, (1.6) is Gauss's law and (1.7) is the ideal gas equation. SI units are taken throughout. A complete derivation of these equations can be found in Boyd and Sanderson (1969).

1.2 Ideal MHD Stability Theory

Before looking at MHD stability theory it is important to first look at MHD equilibrium since instability occurs from movement away from an equilibrium. The standard MHD equilibrium equations are

$$\nabla p = \mathbf{j} \times \mathbf{B}, \tag{1.8}$$

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}, \tag{1.9}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.10}$$

It is noted that these equations are just the time independent form of the full MHD equations with $\mathbf{v} = 0$; the equilibria of interest are static.

In a cylindrical coordinate system (R,y,ϕ), these equations under the assumption that $\partial / \partial \phi = 0$ can be reduced to the Grad-Shafranov equation

$$-\Delta^* \psi = R^2 \mu p'(\psi) + I I'(\psi), \tag{1.11}$$

with

$$\triangle^* \equiv R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial y^2}$$

and

$$I(\psi) = RB_\phi$$

where ψ is a stream function (see Bateman 1978).

1.2.1 Linearised Equations

Starting from the ideal MHD equations, all variables are then expressed in terms of their equilibrium value plus an arbitrarily small perturbation, e.g.

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$$

where the subscripts $_0$ and $_1$ represent the equilibrium and perturbed values respectively. The linearised equations (neglecting products of small factors) are

$$\begin{aligned}
\rho_0\frac{d\mathbf{v}_1}{dt} &= -\nabla p_1 + \mathbf{j}_0 \times \mathbf{B}_1 + \mathbf{j}_1 \times \mathbf{B}_0, && (1.12)\\
\frac{\partial \rho_1}{\partial t} &= -\mathbf{v}_1 \cdot \nabla\rho_0 - \rho_1\nabla\cdot\mathbf{v}_1, && (1.13)\\
\frac{\partial p_1}{\partial t} &= -(\mathbf{v}_1\cdot\nabla p_0 + \gamma p_0\nabla\cdot\mathbf{v}_1), && (1.14)\\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla\times(\mathbf{v}_1\times\mathbf{B}_0), && (1.15)\\
\mathbf{j}_1 &= \frac{1}{\mu}\nabla\times\mathbf{B}_1, && (1.16)\\
\nabla\cdot\mathbf{B}_1 &= 0, && (1.17)\\
\frac{p_1}{p_0} &= \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}. && (1.18)
\end{aligned}$$

A useful transformation is to write the equations in terms of a displacement vector $\boldsymbol{\xi}$ where

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{v}_1, \tag{1.19}$$

such that on combining the linearised equations they form a single second order partial differential equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}\{\boldsymbol{\xi}\}, \tag{1.20}$$

where

$$\begin{aligned} \mathbf{F}\{\boldsymbol{\xi}\} = \nabla(\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) + \frac{1}{\mu}(\nabla \times \mathbf{B}_0) \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)] \\ + \frac{1}{\mu}(\nabla \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)]) \times \mathbf{B}_0 \end{aligned} \tag{1.21}$$

is the force operator. The two equations above are the usual starting point for linear stability analysis and can be combined with the boundary condition

$$\xi_\perp = 0, \tag{1.22}$$

to isolate the system. For reference, $\xi_\perp$ is the perpendicular displacement to the equilibrium magnetic field. This is not however the end of the argument over what the boundary conditions should be. The condition $\xi_\perp = 0$ is rigorous enough when solving only force-free systems (as argued by Raadu (1972)). If gas pressure is included in the equations then boundary conditions must also be set up on the parallel component of the displacement; a subject of some previous debate. Hood and Priest (1979) included radial pressure but restricted their choice to $\xi_{||} = 0$ whereas Einaudi and Van Hoven (1981, 1983) included the parallel component of the displacement and suggested that it need not vanish but that it could satisfy the condition that the total energy of the corona is conserved. Their *flow through* conditions are

$$
\begin{aligned}
\xi_{\perp}(0) &= \xi_{\perp}(L) = 0, \\
\xi_{\|}(0) &= \xi_{\|}(L), \\
\frac{\partial \xi_z}{\partial z}(0) &= \frac{\partial \xi_z}{\partial z}(L),
\end{aligned}
\tag{1.23}
$$

where L is the length of loop under consideration. These conditions simplify the manipulation of the energy integral greatly. However, Rosner et. al. (1986) and Cargill et. al. (1986) have argued that the flow through conditions set upon $\xi_{\|}$ and ξ_z are unlikely since they assume that the photosphere can react to perturbations of the corona quickly, but this is not the case and it acts more as a reflecting boundary and hence all three components of $\boldsymbol{\xi}$ should vanish there (the so called *rigid wall* boundary conditions).

There are two main methods of stability analysis, the energy principle being one, the other being normal mode analysis.

1.2.2 The Energy Principle

The energy principle is derived by multiplying equation (1.20) by $\dot{\boldsymbol{\xi}}$ (the time derivative of the displacement vector) and integrating over the volume with equation (1.22) as a boundary condition. Then, using the self-adjointness property of the force operator, we see that

$$\frac{\partial}{\partial t}[\frac{1}{2}\int \rho_0 \dot{\boldsymbol{\xi}}^2 d\mathcal{V} - \frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}\{\boldsymbol{\xi}\}d\mathcal{V}] = 0, \tag{1.24}$$

i.e the total perturbed energy is a constant. The potential energy in the above is denoted by

$$\delta W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}\{\boldsymbol{\xi}\}d\mathcal{V}. \tag{1.25}$$

It follows that if δW is negative then the system is linearly unstable, otherwise it is stable, except at $\delta W = 0$ where marginal stability takes place. In depth discussions of the energy principle can be found in Kulsrud (1962, 1964) and Laval, Mercier and Pellat (1965).

1.2.3 Normal Mode Analysis

Normal mode analysis is where the displacement vector $\boldsymbol{\xi}$ is expressed as

$$\boldsymbol{\xi}(r,t) = \boldsymbol{\xi}(r)\exp(-i\omega t), \tag{1.26}$$

such that equation (1.20) becomes

$$-\rho_0\omega^2\boldsymbol{\xi} = \mathbf{F}\{\boldsymbol{\xi}\}, \tag{1.27}$$

an eigenvalue problem, solvable with the appropriate boundary conditions. It is clear that instability occurs for negative eigenvalues ($\omega^2 < 0$), indeed only one negative eigenvalue ensures instablity of the system, but all eigenvalues must be positive ($\omega^2 > 0$) to ensure full stability. The point at which $\omega^2 = 0$ is called the point of marginal stability.

1.3 Basic Papers

The roots of solar coronal stablility theory eminated from theories laid down in the early stages of the nuclear fusion programme. It was Lundquist (1951) who first proposed the form of the energy integral, generalised by Berstein et. al. (1957) for hydromagnetic stability problems in fusion research. Newcomb (1960) took the energy principle and reduced stability investigation to finding the roots of a second order ordinary differential equation.

Anzer (1967) used the generalised form of Bernstein et. al. to look at the storage of energy in solar flares and found that all force-free magnetic fields with cylindrical symmetry were unstable to m=1 kink modes. However, Raadu (1972) used Newcomb's stability criterion and showed that coronal magnetic fields are anchored in the dense photosphere providing a strong stabilising effect known as line-tying.

Connor, Hastie and Taylor (1979) took the energy integral and introduced a perturbation for the localised modes (what they called the ballooning mode approximation) to investigate the stability of axisymmetrical toroidal plasmas at high n modes. They found that the mode structure was determined by higher order eigenvalue equations which related the lower order local eigenvalue to the true eigenvalue of the problem.

Hastie and Taylor (1981) showed that the stability equations for localised (ballooning) modes with shear, developed in the paper by Connor, Hastie and

Taylor (1979) were less valid when the shear of the equilibrium became weak. They confirmed that whilst the growth rate of high n mode instabilities was linear with $1/n$, when weak shear was introduced it became oscillatory at intermediate values of n with an amplitude and period proportional to $1/n^2$. A more in-depth treatment of toroidal systems was given by Dewar and Glasser (1983).

Hood (1986a) used this method to model coronal systems by including line-tying effects. These effects were analysed in greater depth by Hood, Van der Linden and Goossens (1988) who took the stability equations for localised (ballooning) modes and reduced them to a system of one-dimensional equations with derivatives in the direction of the equilibrium field; a one-dimensional system being easier to solve numerically. This allowed them to study the effect of each physical phenomena in relation to each other in greater detail.

Hood, Van der Linden and Goossens (1991) studied magnetic, thermal and coalesced magnetothermal instabilities by using the ballooning approximation to study growth rates. They compared these approximate solutions to numerical solutions of the full linear MHD equations in a one dimensional cylindrical equilibrium. More information was obtained by doing a WKB analysis on the growth rates.

De Bruyne and Hood (1992) looked at the stability of line-tied coronal loops by using a finite Fourier method (based on that of Einaudi and Van Hoven (1981, 1983)). Sufficient conditions for instability were obtained using an extended Suydam criterion, the critical loop length being expressed as a function of the central wave number of the fourier expansion.

Approximate solutions to the critical length of solar coronal loops were found by Hood, De Bruyne, Van der Linden and Goossens (1994) by using a WKB expansion in $1/n$ based on that used in Connor, Hastie and Taylor (1979). Unlike all other solutions found thus far, these only required minimal computing time to evaluate. The solutions were compared to numerical solutions using a finite element/Fourier method (see Van der Linden et al., 1990). The correllation between analytical and numerical solutions found was very good, down to an error of $1/n^2$ for high n modes, which is the error one would expect from a WKB expansion.

Had a fast and easy method of computing magnetic instabilities been found? The work of Hood, De Bruyne, Van der Linden and Goossens (1994) was only for one specific magnetic field: the Gold-Hoyle equilibrium. The method needed to be extended to a more general case.

1.4 Aims of the Thesis

The aim of the work in this thesis is to develop the work of Hood, De Bruyne, Van der Linden and Goossens (1994) to find a simple procedure to model the criticality of a twisted solar coronal loop with a general magnetic field. The coronal loop will be modelled by simplifying it to a cylindrically symmetric loop of plasma with a magnetic field of the form

$$\mathbf{B} = (0\ ,\ B_\theta(r)\ ,\ B_z(r)),$$

satisfying the magnetohydrostatic equation. The marginal stability equations will be solved by a WKB approximation method and analytical solutions to the stability of the loop will be derived. The problem will be split into two halves, namely force-free and non force-free fields and the results compared to numerical solutions of the full equations.

2 WKB Theory

> 'The world was created by a very smart person and it probably took him more than seven days. Probably took him a month. It's that complicated.'
> (Mel Brooks, American Premiere, September 1981)

Although early work was done on the subject by Liouville and Jeffries, it is after Wentzel, Kramers and Brillouin that this mathematical method is named, due to their work on this subject in the 1920's. In fact the WKB method is often reffered to as the WKBJ method. The method was developed to approximate the solutions of second order linear homogeneous differentiable equations that can be written in the standard form

$$y''(x) + q(x;\lambda)y(x) = 0, \quad \lambda \gg 1, \tag{2.1}$$

where a prime denotes differentiation with respect to x.

2.1 Liouville's Problem

Consider the equation

$$y''(x) + [\lambda^2 q_1(x) + q_2(x)]y(x) = 0, \quad \lambda \gg 1. \tag{2.2}$$

Assuming that $q_1(x)$ is differentiable and $q_2(x)$ is continuous, we have that as $\lambda \to \infty$

$$q_1(x)y(x) = 0,$$

a trivial solution and so we need to find an approximation to the solution. Putting $y(x) = \exp(\lambda G(x, \lambda))$, where G is a function that can be expanded in inverse powers of λ, into equation (2.2) gives

$$G'^2 + q_1(x) + \frac{1}{\lambda}G'' + \frac{1}{\lambda^2}q_2(x) = 0. \tag{2.3}$$

G is expressed as a power series in λ^{-1} of the form

$$G(x, \lambda) = G_0(x) + \frac{1}{\lambda}G_1(x) + \frac{1}{\lambda^2}G_2(x), \tag{2.4}$$

and substituting equation (2.4) into equation (2.3) gives

$$(G_0'(x) + \frac{1}{\lambda}G_1'(x) + \ldots)^2 + q_1(x) + \frac{1}{\lambda}(G_0''(x) + \frac{1}{\lambda}G_1''(x) + \ldots) + \frac{1}{\lambda^2}q_2(x) = 0. \tag{2.5}$$

The O(1) and O($\frac{1}{\lambda}$) equations of this are

$$G_0'^2(x) + q_1(x) = 0, \tag{2.6}$$

$$2G_0'(x)G_1'(x) + G''(x) = 0, \tag{2.7}$$

respectively. The solutions to equation (2.6) are obvious, and are

$$G_0(x) = \pm i \int \sqrt{q_1(x)}dx, \quad \text{for } q_1(x) > 0 \tag{2.8}$$

and

$$G_0(x) = \pm \int \sqrt{-q_1(x)}dx, \quad \text{for } q_1(x) < 0. \tag{2.9}$$

Using these solutions in equation (2.7) yields

$$G_1(x) = -\ln \sqrt{G_0'}. \tag{2.10}$$

Substituting back into the form for y, gives

$$y(x) \approx \frac{c_1 \cos\left[\lambda \int \sqrt{q_1(x)}dx\right] + c_2 \sin\left[\lambda \int \sqrt{q_1(x)}dx\right]}{\sqrt[4]{q_1(x)}}, \quad \text{for } q_1(x) > 0, \quad (2.11)$$

$$y(x) \approx \frac{c_1 \exp\left[\lambda \int \sqrt{-q_1(x)}dx\right] + c_2 \exp\left[\lambda \int \sqrt{-q_1(x)}dx\right]}{\sqrt[4]{-q_1(x)}}, \quad \text{for } q_1(x) < 0, \quad (2.12)$$

where c_1 and c_2 are arbitrary constants. Equation (2.11) is called the right hand WKB approximation (since it is valid for the region on the right hand side of the turning point) and equation (2.12) is the left hand WKB approximation for similar reasons. Unfortunately, these approximations are not valid at (or near to) the zeros of $q_1(x)$. These points are known as turning points and will be discussed in the chapter.

2.2 The Airy Equation

In many instances the equation being solved can be reduced to the generalised Airy equation,

$$\frac{d^2y}{dx^2} + xy = 0, \tag{2.13}$$

whose solution is

$$y = c_1 Ai(x) + c_2 Bi(x), \tag{2.14}$$

where Ai & Bi are the Airy functions of the first and second kind respectively and c_1 & c_2 are arbitrary constants. These functions are well known and their integral representations are

$$\begin{aligned} Ai(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)dt, \\ Bi(x) &= \frac{1}{\pi}\int_0^\infty [\exp\left(-\frac{1}{3}t^3 + xt\right) + \sin\left(\frac{1}{3}t^3 + xt\right)]dt. \end{aligned}$$

They can also be related to Bessel functions.

2.3 Turning Point Problems

The simplest turning point problem is when $q_1(x)$ has only one zero, since these are the easiest to solve. However, scientific research rarely exhibits such simplicity and the most common turning point problem that is solved is that when $q_1(x)$ has two zeros. Consider the equation

$$y''(x) - \lambda^2 q(x)y(x) = 0, \quad y(\pm\infty) = 0, \tag{2.15}$$

where $q(x)$ has two simple turning points at $x = a$ and $x = b, (a < b)$: we require to find the eigenvalue λ. Investigating the intervals, let us assume

$$q(x) > 0, \quad x < a \quad \text{(exponential solution)}, \tag{2.16}$$

$$q(x) < 0, \quad a < x < b \quad \text{(oscillatory solution)}, \tag{2.17}$$

$$q(x) > 0, \quad x > b \quad \text{(exponential solution)} \tag{2.18}$$

and generate four WKB solutions y_1, y_2, y_3, y_4 with y_1 satisfied within the interval $x < a$ matched onto y_2 in $a < x < b$ and y_3 satisfying $x > b$ matched onto y_4 in $a < x < b$. Meaning that finally y_2 must equal y_4. So for $x < a,\ y(-\infty) = 0$ we have

$$y_1(x) = \frac{c_1 \exp\left[-\lambda \int_x^a \sqrt{q(u)}du\right]}{\sqrt[4]{q(x)}}, \tag{2.19}$$

which matches to the solution

$$y_2(x) = \frac{2c_1 \sin\left[\lambda \int_a^x \sqrt{-q(u)}du + \frac{\pi}{4}\right]}{\sqrt[4]{-q(x)}}, \quad a < x < b, \tag{2.20}$$

and we have for $x > b,\ y(+\infty) = 0$

$$y_3(x) = \frac{c_2 \exp\left[-\lambda \int_b^x \sqrt{q(u)}du\right]}{\sqrt[4]{-q(x)}}, \tag{2.21}$$

which matches to

$$y_4(x) = \frac{2c_2 \sin\left[\lambda \int_x^b \sqrt{-q(u)}du + \frac{\pi}{4}\right]}{\sqrt[4]{-q(x)}}, \quad a < x < b. \tag{2.22}$$

Therefore in $a < x < b,\ y_2 = y_4$ if and only if

$$\lambda \int_a^b \sqrt{-q(u)}du = (n + \frac{1}{2})\pi \text{ and } c_1 = (-1)^n c_2, \tag{2.23}$$

where n is the number of radial nodes. Equation (2.23) is known as the *Bohr - Sommerfeld* condition, (see Bender and Orszag 1978). A more in depth discussion of WKB can be found in Nayfeh (1981).

3 WKB Estimates for the Onset of Ideal MHD Instabilities in Solar Coronal Loops

Anything that happens, happens.
Anything that in happening, causes something else to happen, causes something else to happen, happens.
Anything that in happening, causes itself to happen again, happens again.
It doesn't necessarily do it in chronological order though.
(From 'Mostly Harmless', by Douglas Adams)

The stability analysis of solar coronal loops is severely complicated by the line-tying effect of the dense photosphere. Even for one-dimensional loops the problem is strictly two dimensional and, normally a large computational effort is required to obtain stability properties of any equilibrium. As mentioned before, Hood, De Bruyne, Van der Linden and Goossens (1994) put forward a new, more efficient way of estimating the critical length of a twisted coronal loop. Since the work covered in the paper is the foundation to the work in this thesis, this chapter will be given over to describing the methods used.

Starting from the marginal stability equations

$$\nabla(\boldsymbol{\xi}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{\xi})+\frac{[\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))]}{\mu}\times\mathbf{B}+\frac{\nabla\times\mathbf{B}}{\mu}\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})]=0, \quad (3.1)$$

where

$$\mathbf{B}=(0\ ,\ B_\theta(r)\ ,\ B_z(r)),$$

they expanded the coronal displacement vector $\boldsymbol{\xi}$ in terms of its radial (ξ_r), parallel (η) and perpendicular (ζ) components and reduced the system of equations to

$$\mathcal{L}\zeta = \mathcal{M}\xi_r' + \mathcal{N}\xi_r + \frac{\gamma\mu p}{B^2}\mathcal{M}(\nabla \cdot \boldsymbol{\xi}), \tag{3.2}$$

$$\gamma\mu p\mathbf{B} \cdot \nabla(\nabla \cdot \boldsymbol{\xi}) = 0, \tag{3.3}$$

$$\frac{1}{r}(rB^2\xi_r')' + \{(\mathbf{B}\cdot\nabla)^2 + \frac{2B_zB_z'}{r} + \frac{B_\theta^2 - B_z^2}{r^2}\}\xi_r = (\mathcal{M}\zeta)' - 2\frac{B_\theta}{r}\frac{\partial\zeta}{\partial z} - (\gamma\mu p\nabla\cdot\xi)', \tag{3.4}$$

where the operators $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are defined as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}, \\
\mathcal{M} &= \frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z}, \\
\mathcal{N} &= \frac{1}{r}(\frac{B_z}{r}\frac{\partial}{\partial\theta} + B_\theta\frac{\partial}{\partial z}),
\end{aligned}$$

(see Appendix A for a full derivation of these equations).

The problem is then split into two halves: force-free fields (where the pressure p is zero); and non force-free fields.

3.1 Force-Free Fields

Equations (3.2) and (3.4) are combined (see Appendix B) and on rearranging give

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$
$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2 - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}]\xi_r = 0. \tag{3.5}$$

Proceeding with the force free Gold-Hoyle equilibrium (Gold and Hoyle 1960),

$$B_\theta = \frac{r}{1+r^2}\ ,\ B_z = \frac{1}{1+r^2},$$

they sought a WKB solution of the form

$$\xi_r = \xi(r,z)\exp\left[imS(r,\theta,z)\right], \tag{3.6}$$

where $m \gg 1$ and S satifies

$$\mathbf{B}\cdot\nabla S = 0, \tag{3.7}$$

in a hope to find the critical length at which the azimuthal mode m, first becomes unstable. It was seen that (3.5) could be solved by taking

$$\xi_r = \xi(r)[\exp{(i\frac{2\pi z}{l})} + 1]. \tag{3.8}$$

Thus satisfying the rigid wall conditions,

$$\xi_r = \zeta = 0 \ \text{ at } z = \pm l/2. \tag{3.9}$$

The leading order equation became

$$\frac{d^2\xi}{dr^2} - m^2\{(\frac{1}{r^2}+1) - \frac{2l}{\pi m(1+r^2)} + O(\frac{1}{m^2})\}\xi = 0, \tag{3.10}$$

a two turning point equation.

The *Bohr - Sommerfeld* condition was evaluated giving an expression for the critical length,

$$l_{crit} = 2m\pi + \pi\sqrt{2} + O(m^{-1}). \tag{3.11}$$

This approximation is compared with numerical results in table (1).

Table 1:

Estimates of the critical loop length obtained numerically by Hood et. al. (1994) and the WKB estimate.

m	l_{crit} *(Numerical)*	l_{crit} *(WKB)*
1	8.13985	10.726
2	15.9982	17.009
3	22.696	23.292
4	29.316	29.576
20	130.38	130.11
30	193.38	192.94
40	256.13	255.77
50	318.97	318.60

As can be seen the agreement at high values of m is very good indeed.

3.2 Non Force-Free Fields

The pressure terms are now included and the final equation for ξ_r becomes

$$\begin{aligned}\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r' \\ +[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 - \frac{2p'}{r}\mathcal{L}\frac{\partial^2}{\partial z^2} + \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{\mathbf{B}\cdot\nabla(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} - B_z\frac{\partial}{\partial z})\} + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2]\xi_r \\ +2\frac{B_\theta}{rB^4}(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})^3\frac{\partial}{\partial z}(\gamma\mu p\nabla\cdot\boldsymbol{\xi}) = 0.\end{aligned} \tag{3.12}$$

Proceeding with the generalised Gold-Hoyle equilibrium,

$$B_\theta = \frac{r}{1+r^2}\ ,\ B_z = \frac{\lambda}{1+r^2}\ ,\ \mu p = \frac{(1-\lambda^2)}{2(1+r^2)^2},$$

they took

$$\xi_r = \xi_r(r,z)\exp im(\theta - z/\lambda) \tag{3.13}$$

and non-dimensionalised the equations by taking $z = l\bar{z}$. The method of Connor, Hastie and Taylor (1979) was then followed by setting

$$\begin{aligned} x &= m^{1/2}(r - r_0), \\ l &= l_0 + \frac{l_1}{m} \end{aligned}$$

and expanding the unknown functions in inverse powers of $m^{1/2}$ thus

$$\begin{aligned} \xi_r &= \xi_0(x,\bar{z}) + \frac{1}{m^{1/2}}\xi_1(x,\bar{z}) + \frac{1}{m}\xi_2(x,\bar{z}) + \ldots, \\ \eta &= \eta_0(x,\bar{z}) + \frac{1}{m^{1/2}}\eta_1(x,\bar{z}) + \frac{1}{m}\eta_2(x,\bar{z}) + \ldots \end{aligned}$$

Equations (3.12) and (3.3) could then be expanded in subsequent powers of $m^{1/2}$ by using equation (3.2) and the definition of $\nabla \cdot \boldsymbol{\xi}$ to eliminate ζ. The various order equations were then solved and an expression for the critical length reached, being

$$l = l_0\left(1 + \frac{1}{m}\frac{rB_z}{2B}\sqrt{\frac{l_0''}{l_0}}\right), \tag{3.14}$$

where l_0 and l_0'' are calculated from the leading order equations. Table (2) compares this approximation to numerical values obtained by Hood et. al. (1994) using a finite element/Fourier code.

Table 2:

Estimates of the critical loop length obtained numerically by Hood et. al. (1994) and the WKB estimate for increasing m, when $\lambda = 0.6$.

m	l_{crit} *(Numerical)*	l_{crit} *(WKB)*
1	3.755	4.3339
2	3.747	3.8542
3	3.660	3.6943
4	3.606	3.6144
5	3.452	3.5664
10	3.452	3.4705
20	3.410	3.4225
50	3.390	3.3937
100	3.383	3.3842
∞	3.376	3.3746

As can be seen the agreement is again very good. They concluded from these results that, given the simplicity of the WKB equations, a rapid test on the stability of any equilibrium could be readily obtained.

4 Extension to the General Case

'Try not. Do or do not.
There is no try.'
(Yoda, from 'The Empire Strikes Back', Lucasfilm Ltd. 1980)

As mentioned in the previous chapter, a new more rapid test on the stability of equilibria had been found but had only been tested on one equilibrium profile, namely that of Gold and Hoyle. In order to expand the method to a more general case some sort of check will need to be applied to any results found to see if what is hoped is true, is actually true. Therefore here, we will apply the method to two more equilibrium profiles. The first profile chosen was the Anzer equilibrium, whose generalised form is

$$\begin{aligned} B_\theta &= r\exp(-r/2), \\ B_z &= \lambda\sqrt{(\sigma + (2+2r-r^2)\exp(-r))}, \\ \mu p &= \frac{(1-\lambda^2)}{2}(\sigma + (2+2r-r^2)\exp(-r)), \end{aligned}$$

where $\lambda = 1.0$ for force-free, $\lambda = 0.6$ for non force-free and $\sigma = 0.15$ always.

The second profile chosen was a new equilibrium with hardly any previous investigative work carried out on it. In generalised form it is

$$\begin{aligned} B_\theta &= r(1-r^2), \\ B_z &= \lambda\sqrt{(1+\frac{1}{3}(1-r^2)^2(1-4r^2))}\ ,\ r \leq 1, \end{aligned}$$

$$
\begin{aligned}
B_\theta &= 0\ ,\ B_z = \lambda\ ,\ r \geq 1, \\
\mu p &= \frac{(1-\lambda^2)}{2}(1 + \frac{1}{3}(1-r^2)^2(1-4r^2)),\ r \leq 1, \\
&= \frac{(1-\lambda^2)}{2},\ r \geq 1,
\end{aligned}
$$

with the same values imposed upon λ.

Graphs of both the above profiles are shown in figure (1) and (2) for $\lambda = 0.6$.

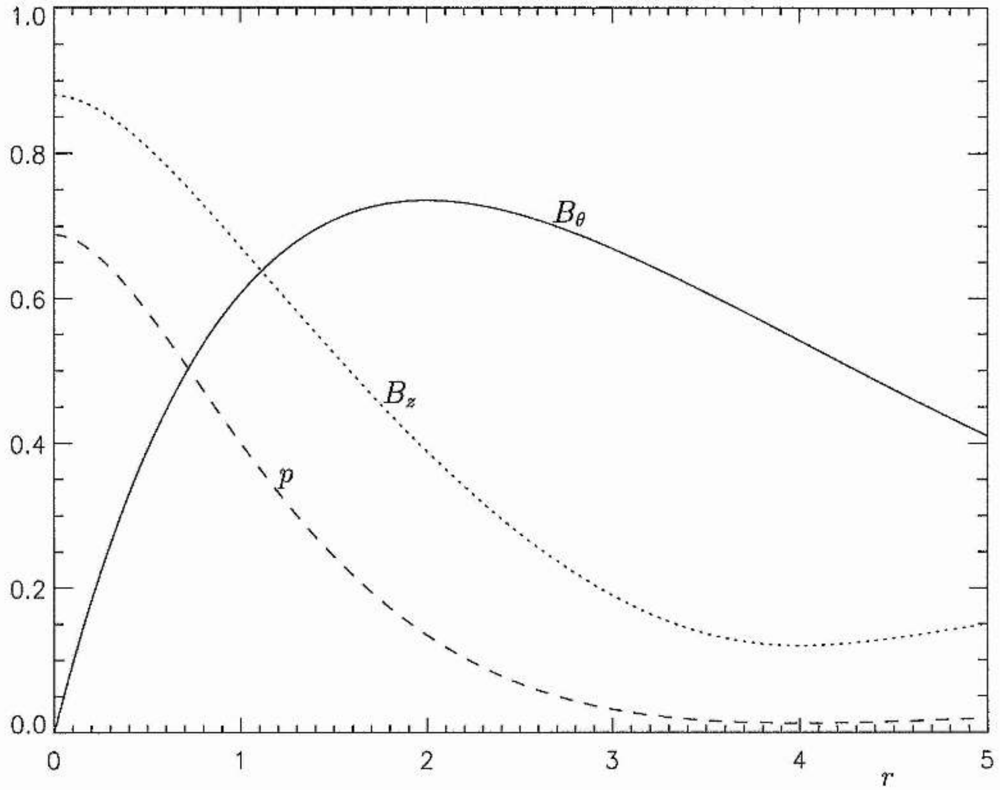

Figure 1: Graph showing the profiles of B_θ (solid line), B_z (dotted line) and p (dashed line) for the Anzer equilibrium when $\lambda = 0.6$.

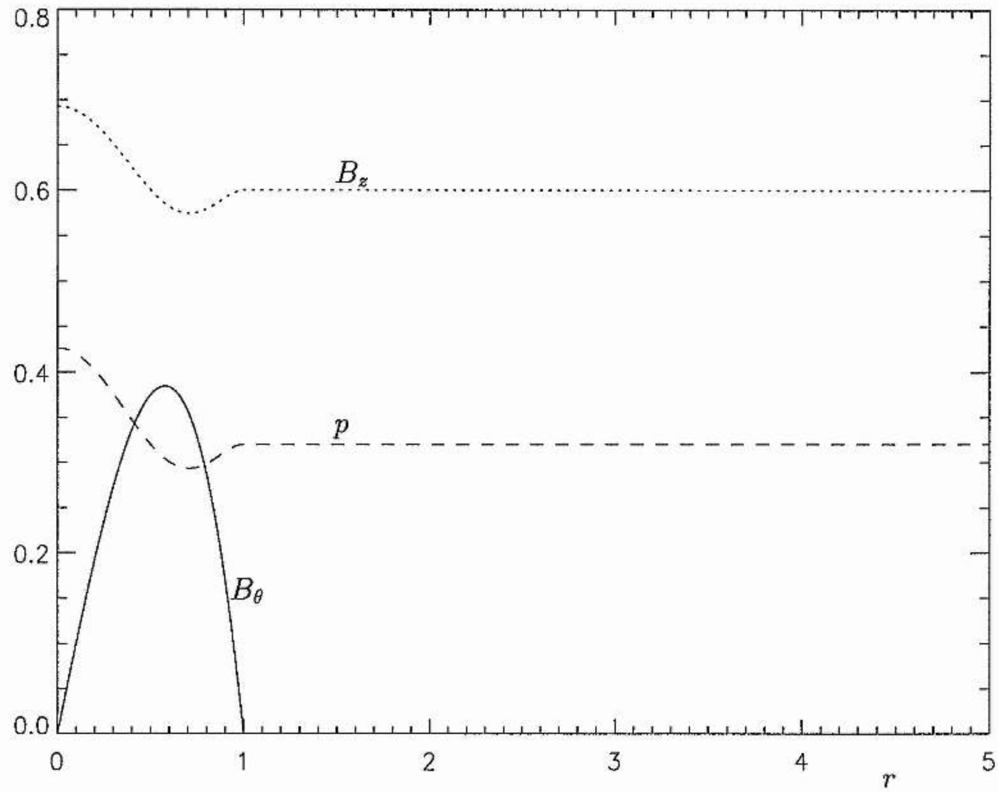

Figure 2: Graph showing the profiles of B_θ (solid line), B_z (dotted line) and p (dashed line) for the new equilibrium when $\lambda = 0.6$.

4.1 Numerical Solution of the Full Equations

As mentioned above, any results found using a WKB approximation will need to be ratified in some way. This is done by comparing the approximate solutions to numerical solutions of the full equations.

The MALTS (MArginal Line Tying Stability) code is used to calculate numerical solutions to the full MHD equations. It is an extension of the POLLUX (Program On Line-tied Loops Under EXcitation) code (see Van der Linden et. al. 1990) using a finite element/Fourier method to solve the equations for velocity to reach an expression for the loop length.

The standard POLLUX equations are reduced to a one dimensional system of the form

$$A \cdot X = \epsilon B \cdot X, \tag{4.1}$$

where ϵ is the aspect ratio of the plasma cylinder. The components of the equations are then written as elements of the matrices A and B. The pressure, temperature, parallel displacement and perpendicular displacement are written as quadratic elements and cubic elements are used for the radial displacement. The above equation is then solved by feeding the program with a central wave number, upper and lower radial bounds and an initial guess to the solution. The method of inverse vector iteration is used to calculate the exact solution.

4.2 Basic Principles

As before, we start with the linearized, ideal, marginal stability equations

$$\nabla(\boldsymbol{\xi}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{\xi})+\frac{[\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))]}{\mu}\times\mathbf{B}+\frac{\nabla\times\mathbf{B}}{\mu}\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})]=0, \quad (4.2)$$

where $\boldsymbol{\xi}$, p and $\mathbf{B}$ are the usual Lagrangian coronal displacement, the equilibrium gas pressure and magnetic field respectively. We consider a cylindrically symmetric equilibrium loop of length l with magnetic field and gas pressure taking the general forms

$$\mathbf{B}=(0,B_\theta(r),B_z(r)), \quad p=p(r), \quad (4.3)$$

satisfying the magnetohydrostatic equation. Again, we reduce the marginal stability equations to

$$\mathcal{L}\zeta=\mathcal{M}\xi_r'+\mathcal{N}\xi_r+\frac{\gamma\mu p}{B^2}\mathcal{M}(\nabla\cdot\boldsymbol{\xi}), \quad (4.4)$$

$$\gamma\mu p\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})=0, \quad (4.5)$$

$$\frac{1}{r}(rB^2\xi_r')'+\{(\mathbf{B}\cdot\nabla)^2+\frac{2B_zB_z'}{r}+\frac{B_\theta^2-B_z^2}{r^2}\}\xi_r=(\mathcal{M}\zeta)'-2\frac{B_\theta}{r}\frac{\partial\zeta}{\partial z}-(\gamma\mu p\nabla\cdot\xi)', \quad (4.6)$$

where $\nabla\cdot\boldsymbol{\xi}$, $\mathbf{B}\cdot\nabla$ and the operators $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ are as defined previously and, in concordance, the photospheric boundary conditions remain the same. A prime denotes differentiation with respect to r.

4.3 Force-Free Fields

Take $p\equiv 0$ so that the final equation for ξ_r is

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$

$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2 - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}]\xi_r = 0. \quad (4.7)$$

Now assume that ξ_r takes the form

$$\xi_r = \xi_r(r,z)\exp[im(S(r) + \theta - \Phi z)], \quad (4.8)$$

where

$$\Phi = \frac{B_\theta}{rB_z} \quad (4.9)$$

represents the twist of the magnetic field and is assumed to be differentiable. This is where the method begins to differ from that of Hood et. al. (1994) since for the force-free Gold-Hoyle equilibrium the twist profile Φ remains a constant and hence on differentiation becomes zero. This fact alone makes the solution to the resulting equation for ξ_r that much easier. Substituting equation (4.8) as a trial solution into equation (4.7) it reduces in leading order to

$$\frac{\partial}{\partial z}\{[\frac{B^2}{r^2B_z^2} + (S' - \Phi' z)^2]\frac{\partial\xi_r}{\partial z} - \frac{4iB_\theta^3}{r^3mB_zB^2}\xi_r\} = 0, \quad (4.10)$$

(see appendix C for a derivation),which on investigation reduces to the equation found for the Gold-Hoyle equilibrium if the appropriate parameters are substituted into the equation.

To simplify the writing of this equation we denote

$$a^2 = \frac{B^2}{r^2B_z^2},\ c = \frac{4B_\theta^3}{r^3B_zB^2}. \quad (4.11)$$

It is easy to see that equation (4.10) can be solved by integrating once and then using an integrating factor to solve the resulting first order differential equation. This yields a solution for ξ_r of

$$\xi_r = A + B\exp[\frac{ic}{m\Phi' a}\arctan(\frac{S' - \Phi' z}{a})] \tag{4.12}$$

where A and B are arbitrary constants. To find an expression for the critical length we are required to find a condition on which this solution depends, as on its own this solution does not tell us anything about the stability of the equilibrium in question. This can be achieved by attempting to find an expression for $S'(r)$ and then evaluating the $Bohr - Sommerfeld$ condition. Applying the rigid wall boundary conditions $\xi_r = 0$ at $z = \pm l/2$, an appropriate expression for $S'(r)$ can be found. It is

$$S'^2 = \Phi'^2 \frac{l^2}{4} - a^2 + \frac{a\Phi' l}{\tan(m\Phi' a\frac{2\pi}{c})}. \tag{4.13}$$

Our solution is based on the assumption that the critical length behaves like

$$l = m\bar{l} = m(l_0 + \frac{l_1}{m}),$$

in other words that as $m \to \infty$, $l/m \to l_0$, a constant value. Since l is a function of r we take that $l_0 = l(r_0)$ and in order to find an expression for this quantity we expand the coefficients of equation (4.13) thus

$$\begin{aligned} x &= m(r - r_0), \\ m\Phi' &= m\Phi''(r_0)(r - r_0) = \Phi_0'' x, \\ a &= a_0 + a'(r_0)\frac{x}{m} + \ldots, \end{aligned}$$

$$c = c_0.$$

Applying that at $r = r_0$ (i.e $x = 0$) $S' = 0$, equation (4.13) yields an expression for l_0, namely

$$l_0 = \frac{2\pi a_0^2}{c_0}. \tag{4.14}$$

All that is left to do now is to find an expression for l_1, the correction term of the critical length. We can manipulate the *tan* function in the expression for S' by expanding it as a power series (i.e $\tan\theta = \theta + \frac{\theta^3}{3} + \dots$). Equation (4.13) becomes

$$S'^2 = \frac{l_1 c_0}{2\pi m} - \frac{a_0^4}{3c_0^2}\pi^2 \Phi_0''^2 x^2. \tag{4.15}$$

From equation (2.23) we have that the *Bohr-Sommerfeld* condition is

$$m\int_{r_1}^{r_2} S'(r)dr = (n + \frac{1}{2})\pi \tag{4.16}$$

which on evaluation gives

$$l_1 = \frac{2}{\sqrt{3}}\pi^2 \frac{a_0^2}{c_0^2}\Phi_0''. \tag{4.17}$$

Then, equations (4.14) and (4.17) can be combined to give a full expression for the critical length of the loop, being,

$$l_{crit} = 2\pi m \frac{a_0^2}{c_0}(1 + \frac{\pi \Phi_0''}{\sqrt{3} m c_0}). \tag{4.18}$$

4.4 Force-Free Results

4.4.1 Anzer Equilibrium

To calculate the WKB approximation we need to find the value of r_0. This is done by looking at the profile of Φ (the twist of the equilibrium). Figure (3) shows this profile and that of its derivative.

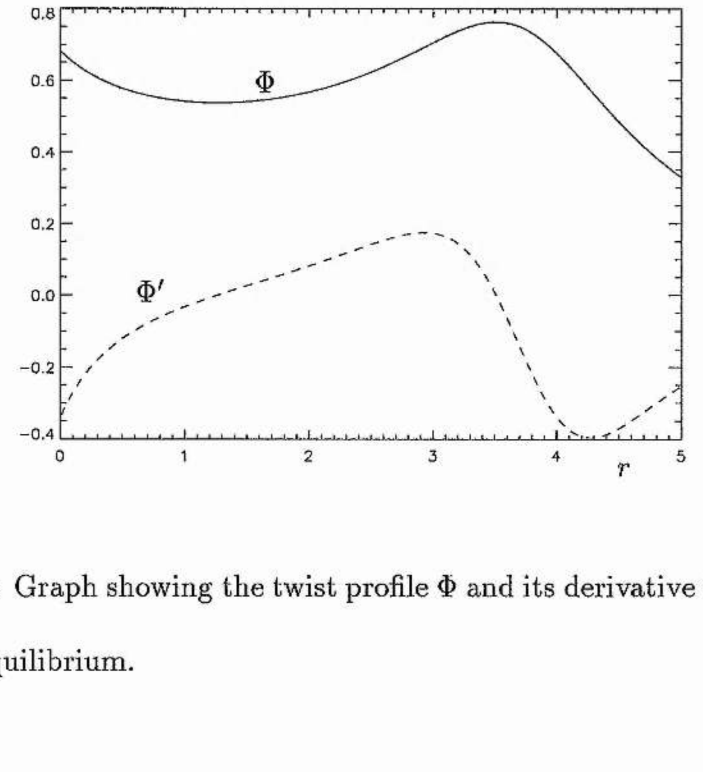

Figure 3: Graph showing the twist profile Φ and its derivative Φ' for the force-free Anzer equilibrium.

As you can see, Φ has a minimum (i.e Φ' has a zero) and this is the point about which Φ is expanded; its equilibrium value. Hence this is the value of r_0. It is 1.26629. Table (3) shows the numerical results for the critical length found by Van der Linden using the MALTS code for increasing Fourier mode number. It shows the convergence of the critical length to a finite solution. Table (4) shows both numerical and WKB approximations to the values of the critical loop length with increasing m for the force-free Anzer equilibrium.

Table 3:

Numerical values for the critical loop length using the force-free Anzer equilibrium with increasing m. The values were found by using the MALTS code based on a finite Fourier series. N_f denotes the number of Fourier modes and $\sigma = 0.15$ throughout.

m	$N_f = 5$	$N_f = 7$	$N_f = 9$	$N_f = 11$	$N_f = 13$	$N_f = 15$
1	8.34283	8.17799	8.09759	8.05095	8.02081	7.99983
2	10.81398	10.67893	10.60775	10.56359	10.53377	10.51241
3	12.64874	12.54826	12.49432	12.46072	12.43779	12.42118
4	12.89734	12.80362	12.75316	12.72162	12.69999	12.68428
5	12.97846	12.88832	12.83978	12.80939	12.78856	12.77345
10	13.06933	12.98425	12.93842	12.90966	12.88996	12.87559
20	13.08997	13.00062	12.96100	12.93273	12.91327	12.89906

As you can see, the agreement between numerically calculated and WKB approximated values of loop length are not all that good. This is puzzling since the numerical solutions exhibit signs that would tend to suggest the WKB results would be more accurate (within the $1/m^2$ error you would expect to find in this sort of analysis) than they are. The reasons for this are as yet unknown.

In calculating the WKB approximation we neglected some terms of the equa-

Table 4:

Numerical values for the critical loop length using the force-free Anzer equilibrium with increasing m, obtained using a finite Fourier series method by Van der Linden, and the WKB estimate of (4.18). An asterisk indicates estimated values and N_f denotes the number of Fourier modes. $\sigma = 0.15$ throughout.

m	l_{crit} *(Numerical)*		l_{crit} *(WKB)*
	$N_f = 15$	$N_f = \infty^*$	
1	7.9998	7.8856	6.9047
2	10.5124	10.3837	10.2068
3	12.4212	12.3172	11.3075
4	12.6843	12.5851	11.8579
5	12.7735	12.6787	12.1881
10	12.8756	12.7838	12.8485
20	12.8991	12.8079	13.1788
∞	12.9042	12.813	13.509

tion, since they were thought to contribute negligible amounts to the final answer. Further investigation (expansion of the full equation - see appendix C) yields an expression for l_0 such that

$$l_0 = \frac{2a^2}{\Phi_0''} \tan\left(\frac{\Phi_0''\pi}{c - \Phi_0''}\right). \tag{4.19}$$

This gives a value for l_0 of 36.07636 which is wrong and so we can rule out any possibility of over-approximation. On the other hand the numerical results have been checked and are still exact, so where the problem lies we do not know.

4.4.2 New Equilibrium

In the case of the new equilibrium a few problems have been encountered. On attempting to find numerical solutions at $\lambda = 1$, MALTS (MArginal Line Tied Stability) had trouble converging to a solution. The program works by feeding it a central wave number (k_0) for the Fourier expansion and an initial guess to the solution. It then uses an iterative procedure to find the exact solution. Finding a value for k_0 for each m value is a matter of guesswork initially but once you have found two (those for $m = 1$ and $m = 2$) approximate values of the others can be calculated since they follow a linear pattern. Values for k_0 at $m = 1$ and $m = 2$ could not be found by trial and error. Since results had already been found for the non force-free case it was taken to expand the value of $\lambda = 0.6$ in the non force-free case up to 1.0 tracing the k_0 value as necessary. Figure (4) shows the loop length as a function of λ as it is traced from the non force-free state to the force-free state.

As you can see the length of loop appears to start to asymptote away from a finite solution. This leads us to suspect that the field may be stable at some values of m. It is clear that $m = 2$ has a finite solution, but do the rest? Figure (5) shows the length as a function of $\ln(1/(1-\lambda))$ which spreads out the right hand end of the graph. This enables us to see exactly what is happening. As you can see, the $m = 2$ line definitely tends towards a finite value but the $m = 3, 4$ & ∞ lines do not. This leads us to deduce that the field is unstable at $m = 1$ & 2 modes and stable for all other modes. The data for $m = 1$ is not shown since the MALTS code continually picked up different harmonics of the solution and hence the results cannot be considered accurate enough.

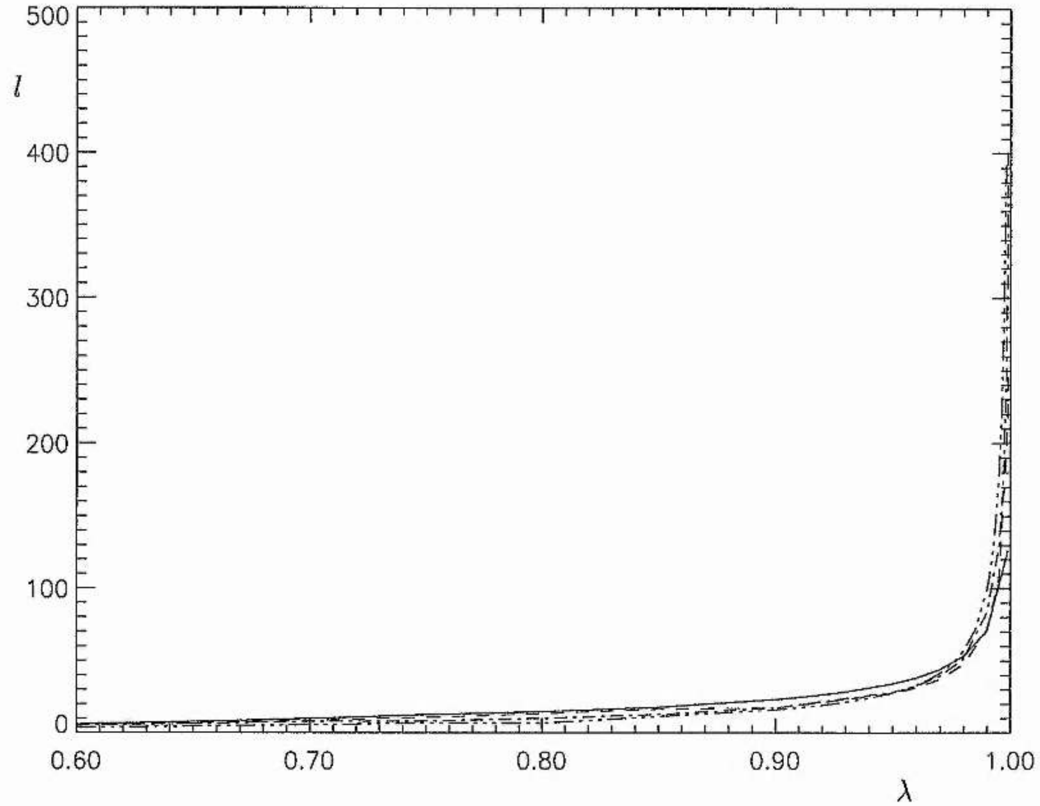

Figure 4: Graph showing loop length versus λ for $m = 2, 3, 4$ & ∞ (in order of decreasing dash length) as λ increases from 0.6 (non force free) to 1.0 (force free).

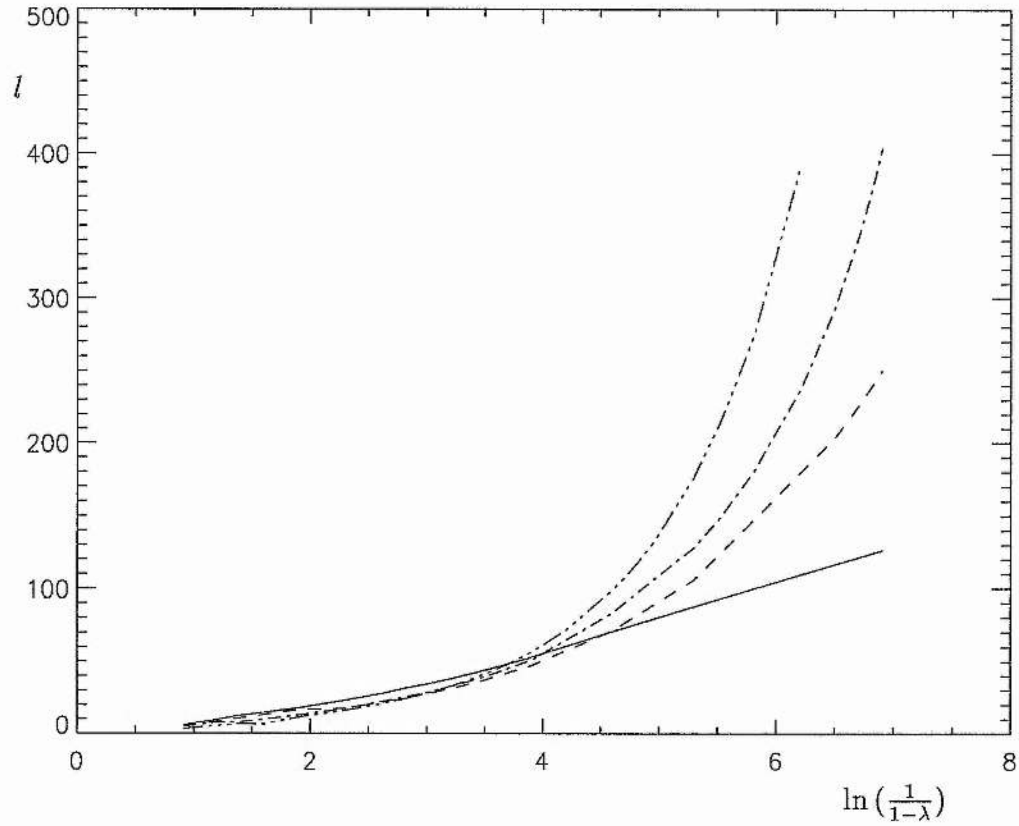

Figure 5: Graph showing loop length versus $\ln\left(\frac{1}{1-\lambda}\right)$ for $m = 2, 3, 4$ & ∞ (in order of decreasing dash length) as λ varies from 0.6 to 1.0.

Unfortunately we have no easy analytical way of proving this hypothesis. We can however put forward some other sufficient conditions of stability that enforce our hypothesis.

Hood and Priest (1979) followed the method of Raadu (1972) in minimising the energy integral by chosing a perturbation of the form

$$\boldsymbol{\xi} = (\xi_r^*,\ -B_z\xi_0^*/B_0,\ B_\theta\xi_0^*/B_0)\cos hz \exp i(m\theta + kz) \tag{4.20}$$

which resulted in the energy integral taking the form

$$\delta W = \frac{1}{\mu}\int dr\{F(\xi_r^{*'})^2 + 2H\xi_r^*\xi_r^{*'} + G\xi_r^{*^2}\}, \tag{4.21}$$

where a prime denotes differentiation with respect to r and

$$\begin{aligned}
F &= \frac{f_1 d_1 r}{f_1 f_2^2 + d_1},\\
H &= \frac{f_1 d_1 - f_1 f_2 d_2}{f_1 f_2^2 + d_1} - 2B_\theta^2,\\
G &= \frac{1}{r}\{\frac{f_1 d_1 + 2 f_1 f_2 d_1 - d_2^2}{f_1 f_2^2 + d_1} + d_3\},\\
f_1 &= \mu\gamma p + B_z^2 + B_\theta^2,\\
f_2 &= krB_\theta - mB_z,\\
d_1 &= B^2(krB_z + mB_\theta)^2 + h^2r^2(\mu\gamma p B_\theta^2 + B^4),\\
d_2 &= 2krB_\theta B^2,\\
d_3 &= h^2r^2B_z^2 + (krB_z + mB_\theta)^2 - 2B_\theta^2 - 2rB_\theta B_\theta'.
\end{aligned}$$

Equation (4.21) can be integrated by parts to give

$$\delta W = \frac{1}{\mu}\int\{F(\xi_r^{*'})^2 + (G - H')\xi_r^{*^2}\}dr \tag{4.22}$$

from which arises a sufficient condition for stability, being

$$G - H' > 0 \tag{4.23}$$

for all r.

Evaluation of this condition for the new equilibrium resulted in $G - H'$ being negative for $m = 1, 2, 3$ and positive for $m = 4$ onwards. This tends to suggest (unlike before) that the mode $m = 3$ is unstable too.

4.5 Non Force-Free Fields

As in Hood, De Bruyne, Van der Linden and Goossens (1994) we take the final equation for ξ_r as

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$
$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 - \frac{2p'}{r}\mathcal{L}\frac{\partial^2}{\partial z^2} + \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{\mathbf{B}\cdot\nabla(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} - B_z\frac{\partial}{\partial z})\} + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2]\xi_r$$
$$+2\frac{B_\theta}{rB^4}(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})^3\frac{\partial}{\partial z}(\gamma\mu p\nabla\cdot\boldsymbol{\xi}) = 0. \tag{4.24}$$

The complication in the solution of this equation is the extra term involving $(\nabla\cdot\boldsymbol{\xi})$. As before we take the form of ξ_r to be $\xi_r = \xi_r(r,z)\exp[im(S(r)+\theta-\Phi z)]$ and expand equations (4.24) and (4.5) to leading order (see Appendix D for a full derivation), giving

$$\frac{\partial}{\partial z}[(a+b(S'-\Phi' z)^2)\frac{\partial\xi_r}{\partial z}] + c\xi_r - d(\frac{\partial\eta}{\partial z}+\xi_r) = 0 \tag{4.25}$$

and

$$\frac{\partial}{\partial z}(\frac{\partial\eta}{\partial z}+\xi_r) = 0, \tag{4.26}$$

where $a = B^2/r^2$, $b = B_z^2$, $c = 2p'B_\theta^2/r^3B_z^2$ and $d = 4\gamma\mu p B_\theta^4/r^4B_z^2(\gamma\mu p + B^2)$. As before we expand x such that

$$x = m^{\frac{1}{2}}(r - r_0)$$

and express the length of the loop in terms of a constant value plus a correction term, thus

$$l = l_0 + \frac{l_1}{m}.$$

We now wish (as in the force free case) to solve for $S'(r)$ in order to be able to use the *Bohr - Sommerfeld* condition as a criterion for stability. This condition arises from standard two turning point WKB problems and so on expressing a form for $S'(r)$ we need to ensure that it has two roots, i.e it needs to be of a quadratic nature. We take it to be

$$S'^2 = \alpha(l - l_0) - \beta(r - r_0)^2. \tag{4.27}$$

To continue in our analysis the values of α and β need to be calculated. If x_1 is the point at which $S'^2 = 0$ then it is seen that

$$S'^2 = \alpha\frac{l_1}{m}(1 - \frac{x^2}{x_1^2}) \tag{4.28}$$

and if l is Taylor expanded in terms of its derivatives (neglecting those of the first order) it can be written as

$$l = l_0 + \frac{l_0''}{2}\frac{x^2}{m}. \tag{4.29}$$

The value of β can now be easily calculated. However, α cannot be calculated so easily. Differentiating equation (4.27) with respect to l gives that $\delta S'^2/\delta l = \alpha$. This will yield an answer for the value of α but to evaluate it we must first calculate l_0. This is done by solving equations (4.25) and (4.26) in a numerical integrator and finding the point at which the minimum length occurs (i.e the value of r_0 such that $l_0 = l(r_0)$). We can now use equation (4.27) to find the correction term in the expression for the critical length. Evaluation of the *Bohr-Sommerfeld* condition gives

$$l_1 = \sqrt{\frac{l_0''}{2\alpha}} \tag{4.30}$$

and so the critical length of the loop is defined by

$$l_{crit} = l_0 + \frac{1}{m}\sqrt{\frac{l_0''}{2\alpha}}. \tag{4.31}$$

The value of l_0'' is now calculated in an numerical integrator by using the expression

$$l_0'' = \frac{l_{-1} - 2l_0 + l_1}{(\delta r)^2} \tag{4.32}$$

where the small change in the radial direction $\delta(r)$ is fixed and $l_{-1} = l(r_0 - \delta r)$ and $l_1 = l(r_0 + \delta r)$.

4.6 Non Force-Free Results

In the case of non force-free fields the value of r_0 is calculated by finding the point at which l_0 reaches a minimum. This is done by setting $S'(r) = 0$ in equation (4.25) and solving it using a numerical integrator. Through detailed calculations (see appendix E) it was realised that the form of the solution should be $(S'^2 + \Phi'^2 z^2)$ and not $(S' - \Phi' z)^2$ as was once thought. Therefore, the results shown hereafter are calculated on this basis.

4.6.1 Anzer Equilibrium

The value of r_0 was found to be 2.61405 and hence approximations to the loop length were calculated. Table (5) shows the numerical results found by Van der Linden for the critical loop length to the non force-free Anzer equilibrium for increasing Fourier mode number. The results show that as the Fourier mode number increases the critical loop length decreases, converging towards a finite solution for all values of m. Table (6) shows the results found numerically by Van der Linden and the WKB estimate to the critical loop length. The agreement of the numerical and WKB results is very good indeed. At $m = 50$ the error is only 0.03%, which is really too good, since the error of WKB is of the order $1/m^2$ and so at $m = 50$ the error would be 0.0004 which is double the error found.

4.6.2 Gold-Hoyle Equilibrium

Out of personal interest I used the methods derived to find approximate lengths for the Gold-Hoyle equilibrium. Table (7) shows the results found in Hood et. al. (1994) numerically and with WKB. They are compared with those found using the above method. As you can see, the two WKB approximations are very similar (equal to three or four decimal places at high orders of m), thus showing that WKB is a strong approximation, especially at high values of m.

Table 5:

Numerical values for the critical loop length using the non force-free Anzer equilibrium with increasing m. The values were found by using the MALTS code based on a finite Fourier series. N_f denotes the number of Fourier modes and $\sigma = 0.15$ throughout.

m	$N_f = 5$	$N_f = 7$	$N_f = 9$	$N_f = 11$	$N_f = 13$	$N_f = 15$
1	3.27003	3.25346	3.24511	3.24005	3.23665	3.23421
2	2.97432	2.93908	2.92274	2.91304	2.90658	2.90195
3	2.92064	2.86397	2.83972	2.82646	2.81800	2.81208
4	2.90260	2.83431	2.80195	2.78479	2.77436	2.76732
5	2.89257	2.81887	2.78170	2.76087	2.74830	2.74004
10	2.86720	2.78572	2.74384	2.71823	2.70100	2.68873
20	2.85378	2.76665	2.72245	2.69573	2.67781	2.66492
50	2.84893	2.75818	2.71156	2.68331	2.66443	2.65096

Table 6:

Estimates of the critical loop length for the non force-free Anzer equilibrium obtained numerically and the WKB estimate of (4.31) for increasing m, when $\lambda = 0.6$ and $\sigma = 0.15$. Estimated values are marked with an asterisk.

m	l_{crit} *(Numerical)* $N_f = \infty^*$	l_{crit} *(WKB)*
1	3.2193	3.0981
2	2.8741	2.8288
3	2.7795	2.7390
4	2.7337	2.6941
5	2.7053	2.6672
10	2.6158	2.6133
20	2.5861	2.5864
50	2.5705	2.5703
∞	2.5601	2.5595

Table 7:

Values for the critical loop length obtained numerically, the WKB estimates for the Gold-Holye field (WKB_1) obtained in Hood et. al. (1993) and (WKB_2) obtained by the methods previously described. $\lambda = 0.6$ and $\sigma = 0.15$ throughtout. Estimated values are indicated by an asterisk.

m	l_{crit} *(Numerical)*		l_{crit} (WKB_1)	l_{crit} (WKB_2)
	$N_f = 13$	$N_f = \infty^*$		
1	3.771	3.775	4.339	4.3265
2	3.757	3.747	3.8542	3.85055
3	3.671	3.660	3.6943	3.6919
4	3.618	3.606	3.6144	3.6125
5	3.584	3.452	3.5664	3.56498
10	3.519	3.452	3.4705	3.46979
20	3.491	3.410	3.4225	3.42219
50	3.473	3.390	3.3937	3.3936
100	3.467	3.383	3.3842	3.38412
∞	3.461	3.376	3.3746	3.3746

4.6.3 New Equilibrium

Table (8) below shows the numerical values for the critical loop length for the new equilibrium. Again they show a convergence towards a finite solution for each value of m as the Fourier mode number increases.

Table 8:

Numerical values of the critical loop length for the non force-free new equilibrium found by using the MALTS code for increasing m and increasing Fourier mode number. $\sigma = 0$ throughout.

m	$N_f = 5$	$N_f = 7$	$N_f = 9$	$N_f = 11$	$N_f = 13$	$N_f = 15$
1	8.05652	7.86071	7.75866	7.71558*	7.71558*	7.71588*
2	6.03471	5.88946	5.81280	5.76552	5.73355	5.71058
3	5.35441	5.22490	5.15674	5.11466	5.08615	5.06561
4	5.01438	4.89136	4.82720	4.78771	4.76097	4.74171
5	4.81194	4.69167	4.62965	4.59164	4.56597	4.54749
10	4.42268	4.30046	4.24106	4.20550	4.18179	4.16487

The values marked with an asterisk are caused by an anomaly in the numerical program MALTS. As described earlier, the program works by inverting a matrix that contains the components of the equations. Hence, if this matrix were to become singular at any time the program will not converge to an exact solution. Due to the way the numerics work, the matrix will become singular if the central wave number is an integer multiple of the solution. In the case of $m = 1$, this happened. The reason for the doubt in the value is, because the program iterates, as soon as it hits this solution it erupts and hence we do not know if this is the true solution or not.

The value found for r_0 was 0.3412 and table (9) shows the comparison between numerically evaluated lengths and the WKB estimate. In this case the MALTS code was again used to find the critical loop length numerically.

Table 9:

Estimates of the critical loop length obtained numerically by using the MALTS code and the WKB estimate of (4.31) for increasing m for the new equilibrium, with $\lambda = 0.6$. Estimated values are marked with an asterisk.

m	l_{crit} *(Numerical)* $N_f = \infty^*$	l_{crit} *(WKB)*
1	7.7156	8.0981
2	5.5704	5.2719
3	4.9392	4.7463
4	4.6231	4.4835
5	4.4341	4.3258
10	4.064	4.0105
∞	3.6939	3.6951

5 Discussion

'This is the greatest work of fiction since vows of fidelity were introduced into the French marriage service.'
(Blackadder Goes Forth, BBC Television Enterprises 1992)

It was found in Hood et. al. (1994) that the WKB estimates for the force-free Gold-Hoyle equilibrium formed a Sturmian sequence for various azimuthal mode numbers. In other words that low m modes have a shorter critical length than high m modes, but the results for the non force-free case formed an anti-Sturmian sequence. The same properties are exhibited here for both the Anzer equilibrium and the new equilibrium.

We have also seen that even though the mathematics is more in depth for the two equilibriums studied here, expressions for the critical length are still obtained using a WKB estimate that are far more simplistic to calculate solutions to than the full numerics. The reason for the more in depth mathematics (as mentioned earlier) is because of the twist profile Φ and its effect on the equations. If we look at the Φ profiles for the three equilibriums they exhibit some interesting but very different properties. Figures (6) and (7) show the twist profiles for the Anzer and new equilibriums. The twist of the Gold-Hoyle equilibrium is of course a constant and so does not need to be drawn.

As you can see the twist profile for the Anzer equilibrium has a minimum at

$r = 1.266$ which is the point about which the coefficients are expanded in

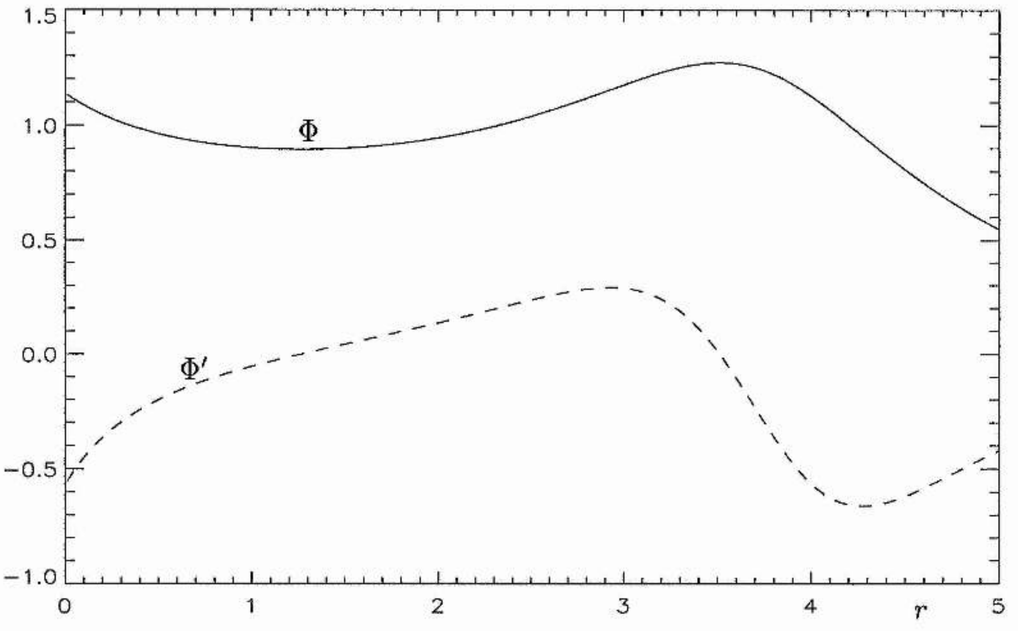

Figure 6: Graph showing the twist profile Φ and its derivative Φ' for the non force-free Anzer equilibrium.

the force free case, but the twist profile for the new equilibrium has no turning points whatsoever. Maybe this is why we have stability in the case of this equilibrium, since neither the Gold-Hoyle or Anzer equilibriums exhibit quite the same properties. There is, however, no discernable connection between the twist profile and the points at which the minimum critical lengths occur for the Anzer and new equilibriums in the non force-free case.

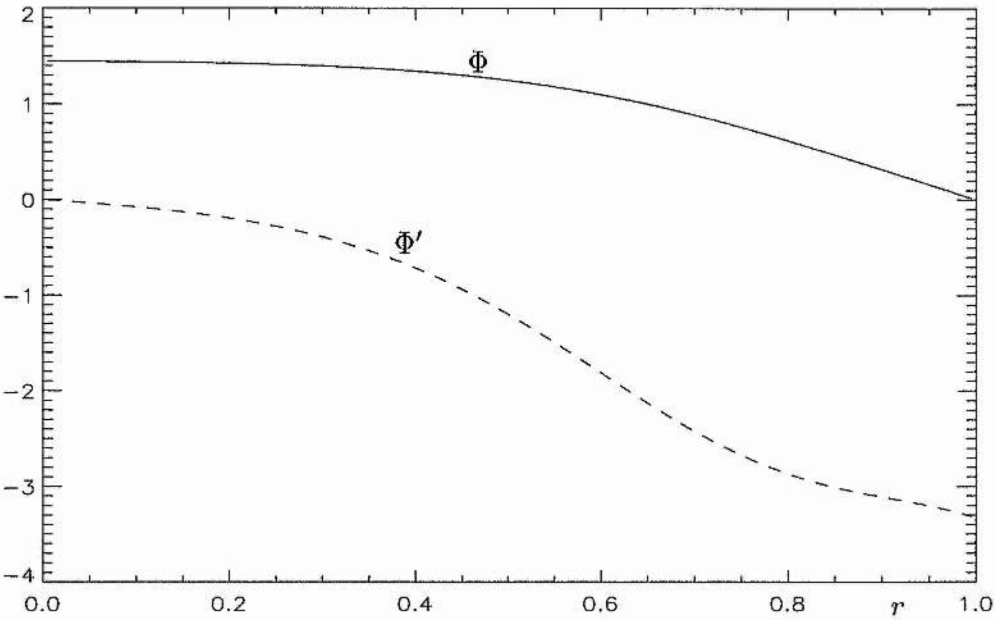

Figure 7: Graph showing the twist profile Φ and its derivative Φ' for the non force-free new equilibrium.

The approximate WKB solutions for the force-free Anzer equilibrium are puzzling. The derivation of the solution is complete, in that we have not over approximated our solution (this is seen since our l_0 is greater than that of the numerical solutions), indeed on expanding the equations fully neglecting terms of $O(m^2)$ or less, the arising solution is greater than that of the approximate one. In order to profer an expanation why, an investigation into the numerical solutions must be done. The numerical program solves the equations by calculating the velocity (which is directly related to the displacement $\boldsymbol{\xi}$) and so by studying the form of the radial velocity component of the numerical solutions some information on the form of the analytical solutions can be gained. Figure (8) shows the radial dependence of the normalised radial velocity component. We see that the plots exhibit a Gaussian style profile that narrows as m increases. This is indeed consistent with our trial solution.

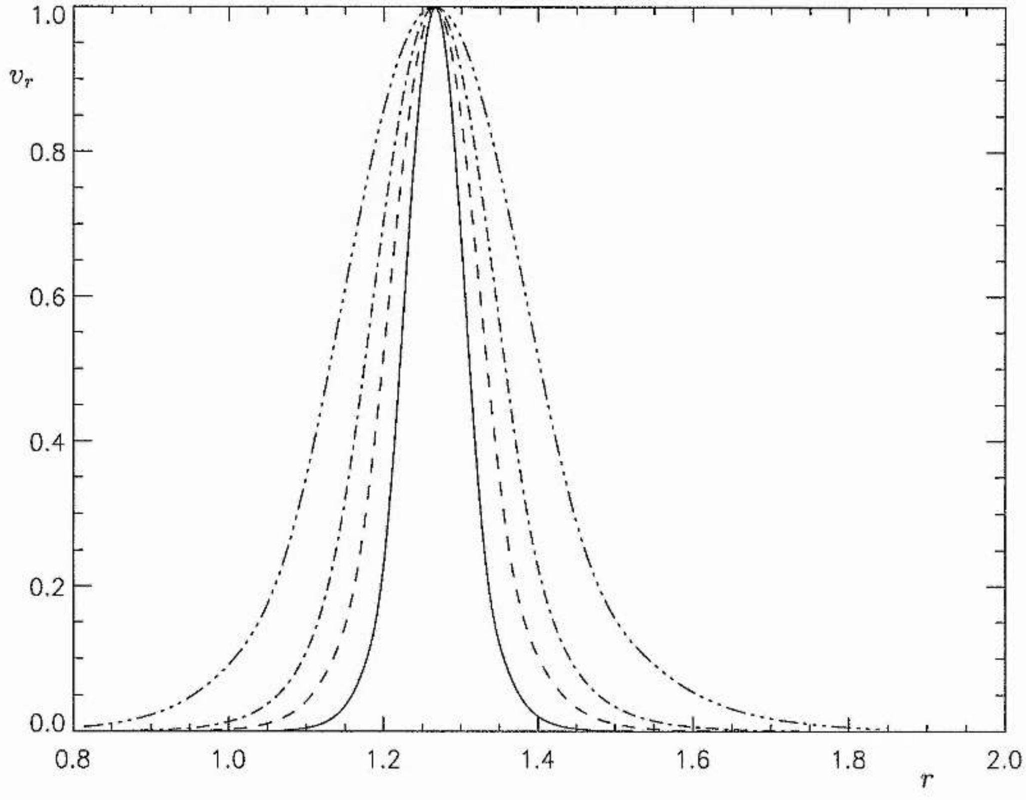

Figure 8: Plots showing the normalised radial velocity component for the numerical solutions of the full equations for $m = 30, 20, 15$ & 10 in order of decreasing dash length respectively.

In reaching an analytical solution we wrote $x = m(r - r_0)$ in order to expand our variables, but this may not be the correct form. If we assume the numerical solution in the radial direction takes the form $v_r = f(r)\exp\left[m^\nu g(r)\right]$ then taking the log of v_r leaves us with $\ln v_r = \ln f(r) + m^\nu g(r)$ and hence plotting the graphs for the different values of m will enable us to see if in fact the scaling is proportional to m or indeed to any other power of m. In figure (9) logarithms have been taken and the radial velocity component plotted for $m = 10, 15, 20$ & 30. The continous line shows $m = 30$ and the other dashed lines show $m = 10, 15$ & 20 scaled according to the above equation as if they were $m = 30$. If with some value of ν for the scaling, the lines lie on top of each other then we will have found the correct power of m to expand the variables by. Looking at figure (9) you can see that although the lines are close they are not the same. They do appear to be moving towards each other at the ends but this is where very small values are being plotted and so limits of resolution will be reached. Figure (10) uses the same idea but the lines are scaled as if the factor is m^2. As you can see the lines match between $z/l = 1.15$ & 1.355 which is the most significant part of the original Gaussian profile (see figure (8)) as the rest just accounts for the 'wings' (the outermost points of the profiles).

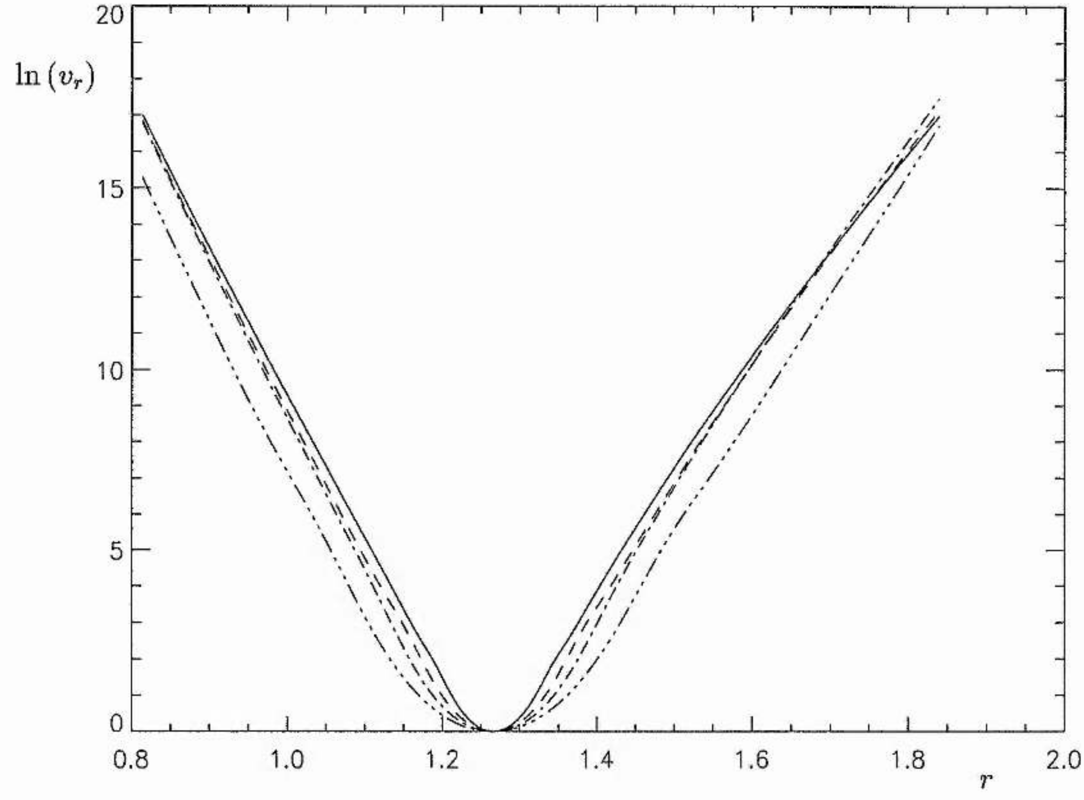

Figure 9: Graph showing plots of the natural logarithm of the radial velocity component for $m = 30, 20, 15$ & 10 in order of decreasing dash length. The lines for $m = 10, 15$ & 20 have been scaled to match that of the $m = 30$ line as if the multiplying factor of the solution is proportional to m.

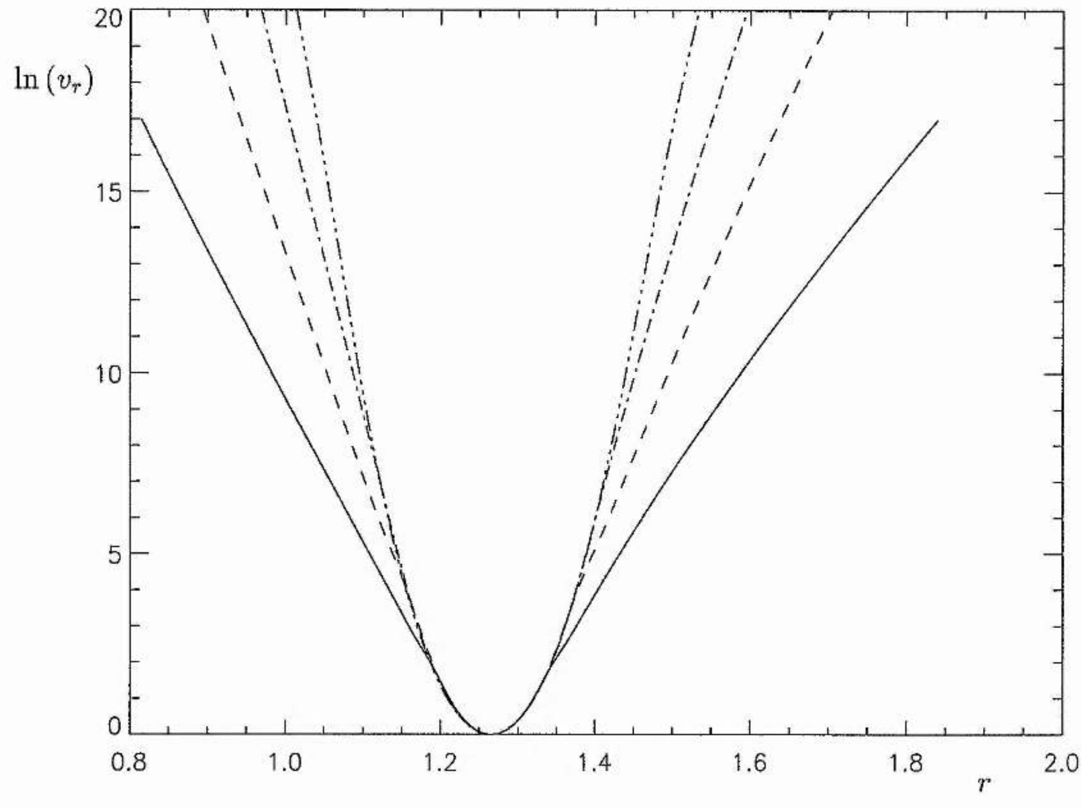

Figure 10: Graph showing plots of the natural logarithm of the radial velocity component for $m = 30, 20, 15$ & 10 in order of decreasing dash length. The lines for $m = 10, 15$ & 20 have been scaled to match that of the $m = 30$ line as if the multiplying factor of the solution is proportional to m^2.

Some of the discrepancies in these graphs could be explained by the additive term of $f(r)$. Therfore, if we write

$$\begin{aligned}\ln v_{r_{30}} &= \ln f(r) + 30^{\nu} g(r),\\ \ln v_{r_{20}} &= \ln f(r) + 20^{\nu} g(r),\end{aligned}$$

then subtracting the second equation from the first gives

$$\ln v_{r_{30}} - \ln v_{r_{20}} = (30^{\nu} - 20^{\nu}) g(r),$$

or

$$\ln (v_{r_{30}}/v_{r_{20}}) = (30^{\nu} - 20^{\nu}) g(r),$$

eliminating the function $f(r)$ and so only leaving a direct dependence on m. Figure (11) shows plots for $\ln (v_{r_{30}}/v_{r_{20}})$, $\ln (v_{r_{20}}/v_{r_{15}})$ and $\ln (v_{r_{15}}/v_{r_{10}})$. In figure (12) these profiles are matched for $\nu = 1$ and in figure (13) they are matched for $\nu = 2$. Again, you can see that the lines matched by a factor of m ($\nu = 1$) do not agree as well as those matched by m^2 ($\nu = 2$). Unfortunately the lines for the m^2 case do not agree all the time and so we cannot definitely say whether or not the scaling should be to m or m^2. The best guess would be m^2.

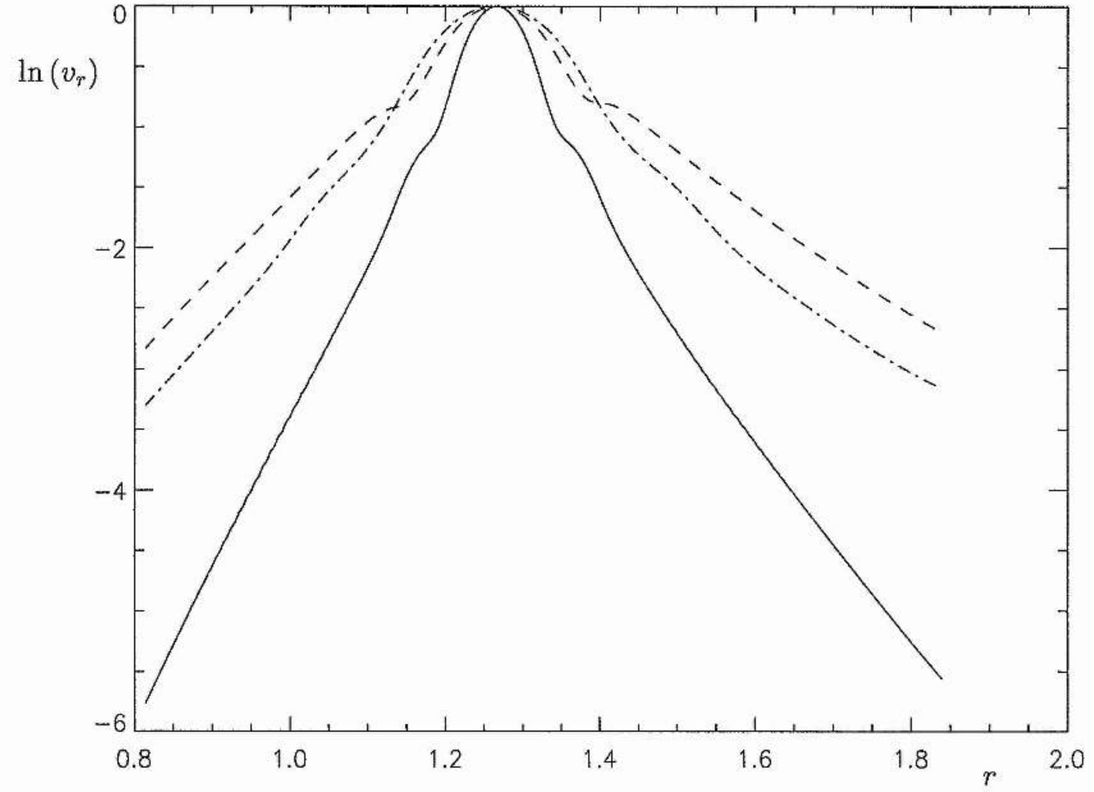

Figure 11: Graph showing plots for $\ln[v_{r_{30}}/v_{r_{20}}]$ (solid line), $\ln[v_{r_{20}}/v_{r_{15}}]$ (dashed line) and $\ln[v_{r_{15}}/v_{r_{10}}]$ (dash-dot line).

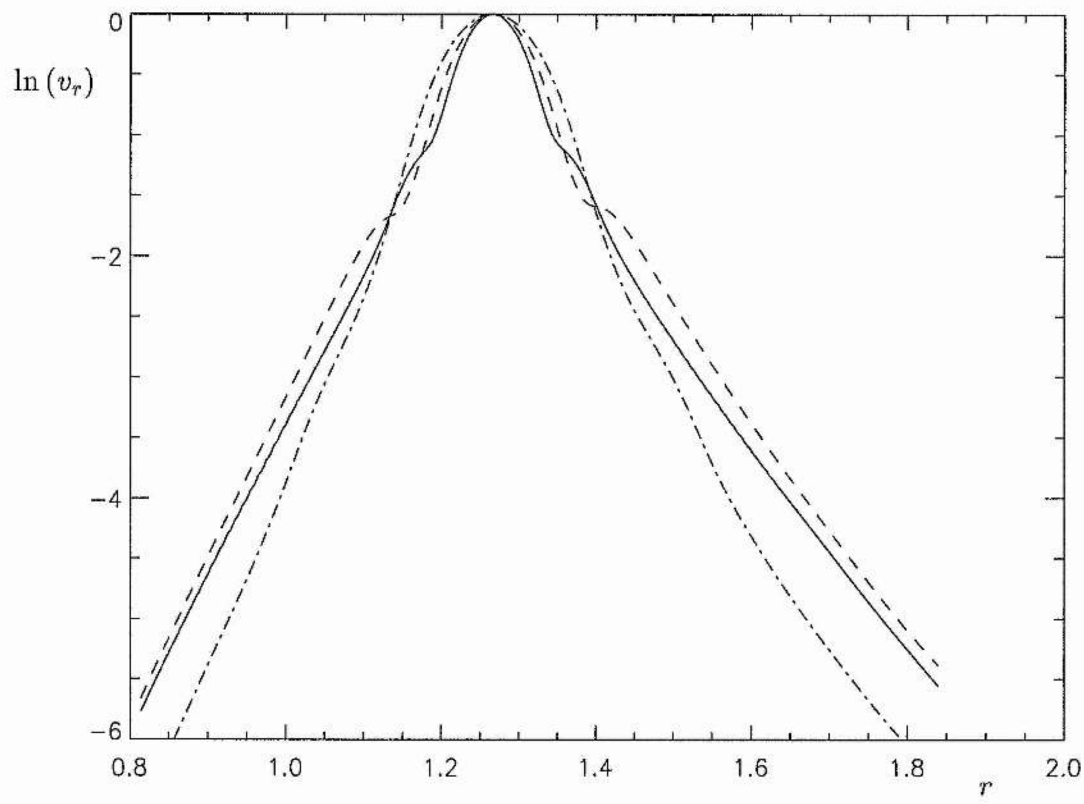

Figure 12: Graph showing plots for $\ln[v_{r_{30}}/v_{r_{20}}]$ (solid line) with $\ln[v_{r_{20}}/v_{r_{15}}]$ (dashed line) and $\ln[v_{r_{15}}/v_{r_{10}}]$ (dash-dot line) scaled to match that of $[v_{r_{30}} - v_{r_{20}}]$ to a factor of m.

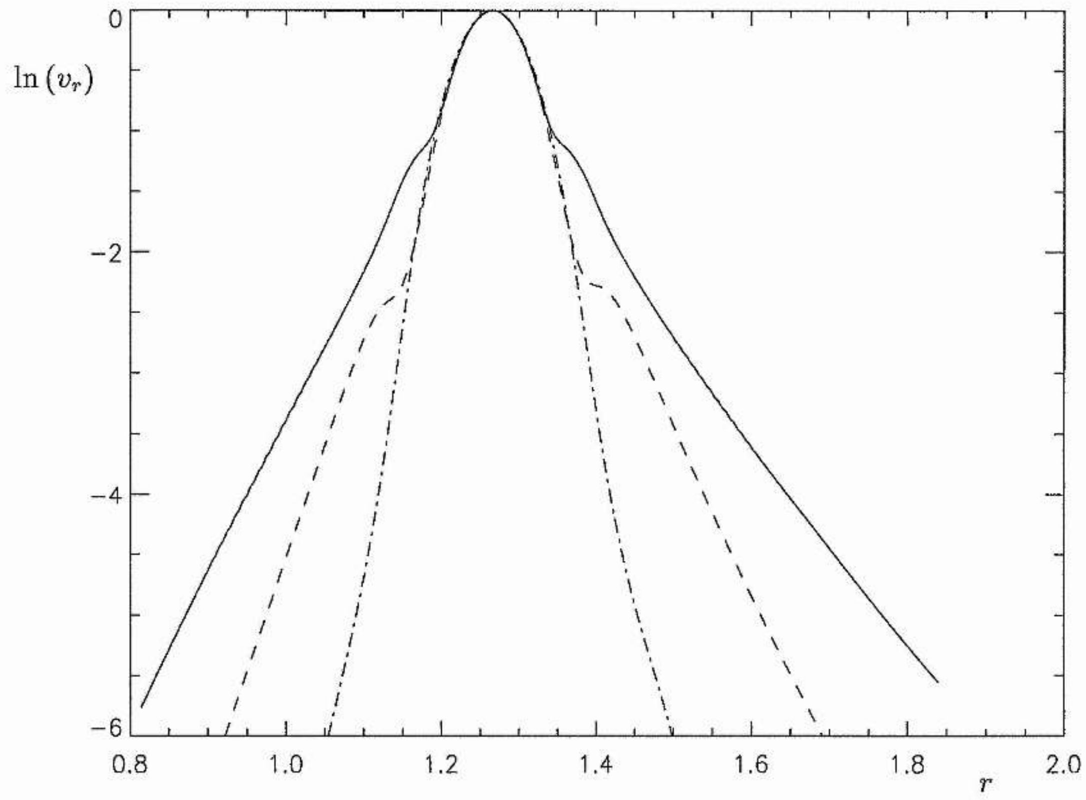

Figure 13: Graph showing plots for $\ln[v_{r_{30}}/v_{r_{20}}]$ (solid line) with $\ln[v_{r_{20}}/v_{r_{15}}]$ (dashed line) and $\ln[v_{r_{15}}/v_{r_{10}}]$ (dash-dot line) scaled to match that of $[v_{r_{30}} - v_{r_{20}}]$ to a factor of m^2.

Figure (14) shows that the numerical solutions for the radial velocity component fit the general profile $\exp[3/8(r-r_0)^2m^2]$ where $r_0 = 1.26629$. The profiles match until they start to fan out at the bottom (the so called wings) and so the question needed to be asked here is should they match at the wings? This would indeed make them true Gaussian profiles. The discrepancies could be explained by 'noise' entering the numerical program since greater errors occur when dealing with smaller numbers. An important point to notice is of course the m^2 factor in the general equation. I have found no equation that can map onto the numerical solutions so readily with only a single m dependence. So, if indeed we can ignore the 'wings' of the profiles (everything from $v_r = 0.2$ downwards) then correspondingly we must ignore this data in all subsequent graphs plotted. The logarithm of 0.2 is 1.61 and so ignoring everything above this mark on the log plots means that we have an m^2 dependence. This is only right since our whole argument here started from something dependent on m^2, but it remains to be shown if anything can explain the data patterns as well. Considering all this, based on a solution that appears quadratic in nature the expansion of x must be $x = m(r - r_0)$. Figure (15) shows the radial velocity plotted as a log against $x = m(r - r_0)$ for various m values. The line $3/8x^2$ is also plotted. As can be seen, all the lines are a near perfect match.

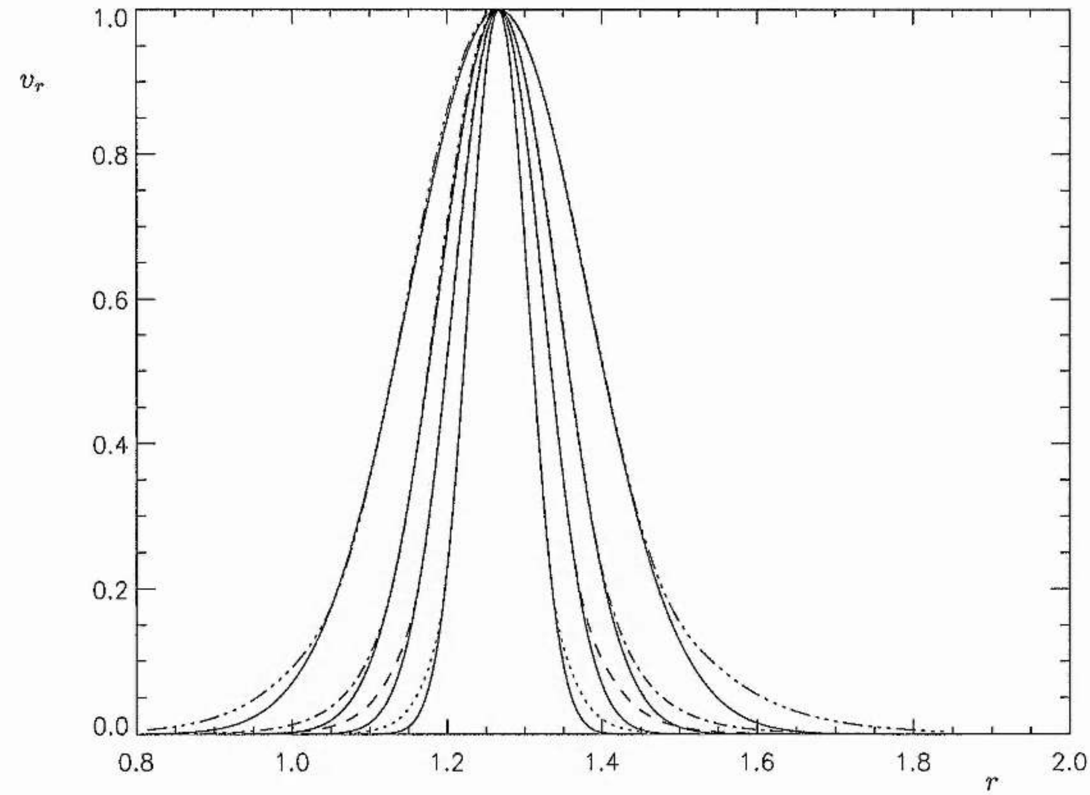

Figure 14: Graph showing plots of the radial velocity component for $m = 30$ (dotted line), $m = 20$ (dashed line), $m = 15$ (dot-dash line) and $m = 10$ (dot-dot-dot-dash line) with the general profile $v_r = \exp\left[3/8(r - r_0)^2 m^2\right]$ drawn over each m value in a solid line.

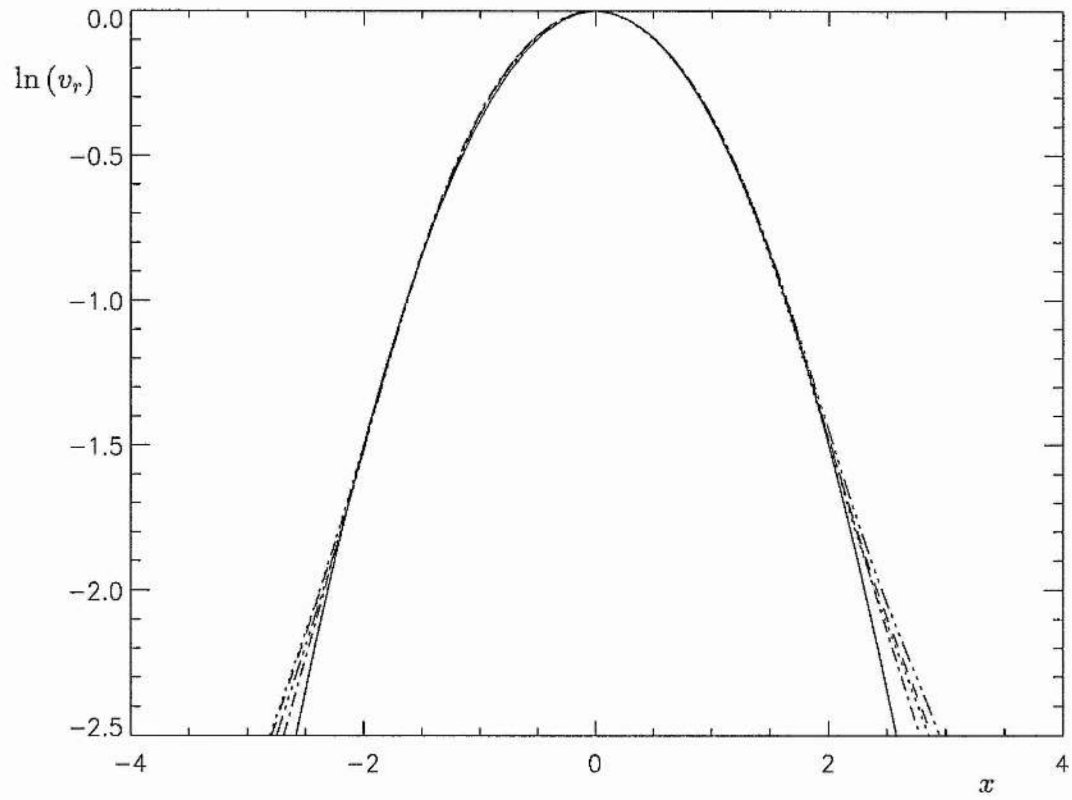

Figure 15: Graph showing normalised logarithmic plots of the radial velocity component for $m = 30$ (dotted line), $m = 20$ (dashed line), $m = 15$ (dot-dash line) and $m = 10$ (dot-dot-dot-dash line) plotted against $x = m(r - r_0)$, along with the solid line $3/8x^2$

Approximating the second differential of the radial velocity component enables us to see where the turning points of the system lie and how they change as m increases. These approximations are shown in figure (16) below.

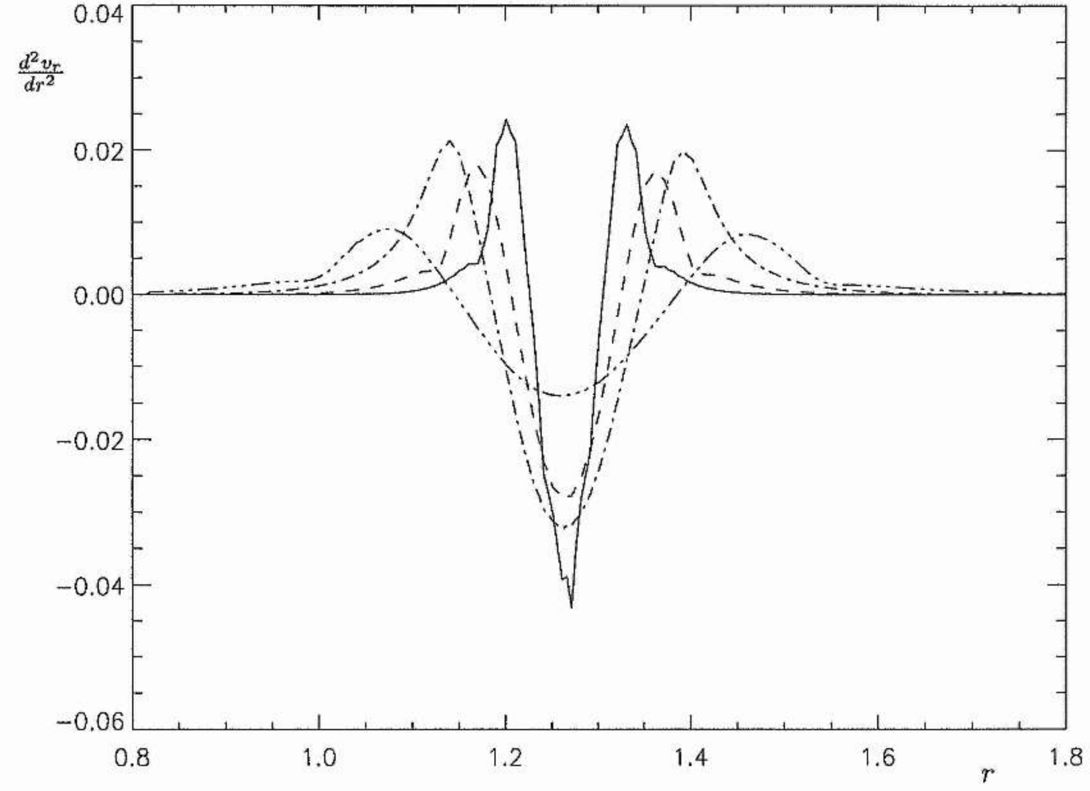

Figure 16: Graph showing plots of the second derivative of the radial velocity component for m=30,20,15 & 10.

Looking at the above figure we see that the positions of the turning points are decreasing in r as m increases, apparently converging to some point (probably $r = 1.26629$). Is the rate of convergence proportional to m^ν? The turning points for each value of m were approximated and various graphs plotted in an attempt to calculate the proportionality, but unfortunately the accuracy to which these points were calculated is not good enough to make a strong argument for or against any m dependence. The second derivative is of course an approximation in its own right.

Looking closely at figures (9), (10), (11), (12) and (13) we see that they show some similar characteristics. They all visibly kink at certain points from what look like quadratic curves to near straight lines for each value of m. This changeover in characteristic is due to the fact that the solutions change from being oscillatory to decaying exponential at the roots of the solutions (the turning points). The solutions are oscillatory between the two roots and exponential either side of them. To the eye it appears as if the points where the graphs kink are different to those of the roots shown in figure (16) but this is simply because the changeover is gradual not instantaneous.

We can also gain some information about the z dependence of the analytical solution by looking at the z dependence of the numerical solution. We assume that the full equations are solvable by taking the form

$$\xi_r = \xi(r)[exp(i\frac{2\pi z}{l}) + 1].$$

The real part of this solution is simply $A(1+\cos\frac{2\pi z}{l})$ where A is a constant. Hence we can plot graphs of this for varying values of m and compare them to plots of the z dependence of the radial velocity component in the numerics. Figure (17) shows these comparisons.

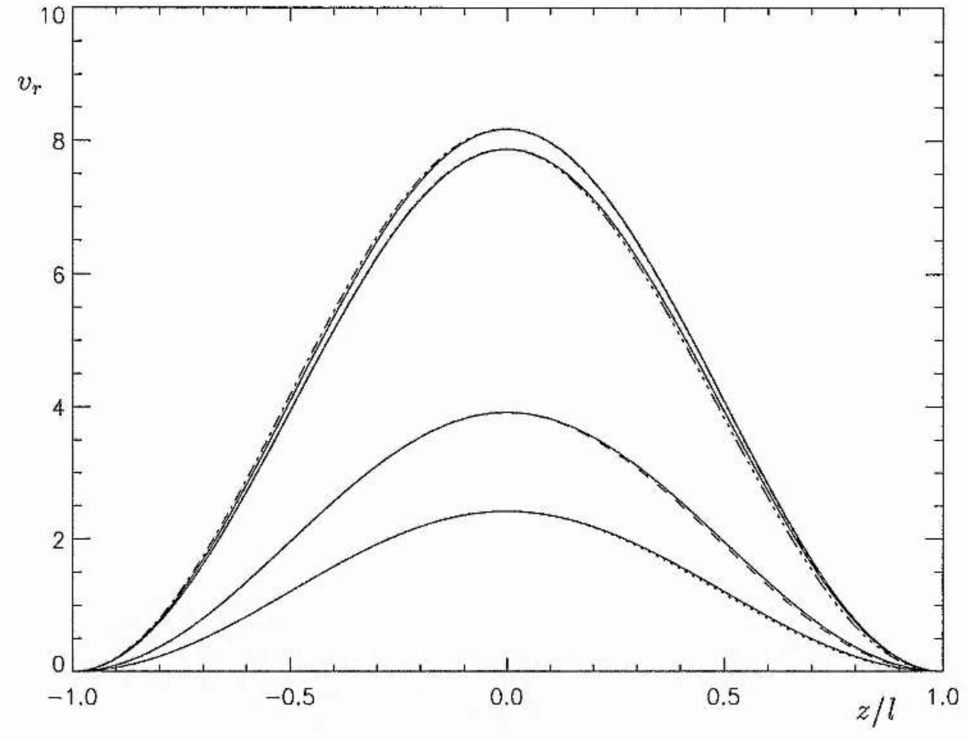

Figure 17: Graph showing the radial velocity component versus z/l of the numerical solution for $m = 30$ (dotted line), $m = 20$ (dashed line), $m = 15$ (dash-dot line) and $m = 10$ (dash-dot-dot-dot line). The corresponding analytical forms are drawn over the top in a solid line for each m value.

The error between the numerical and analytical solutions are shown in figure (18) below.

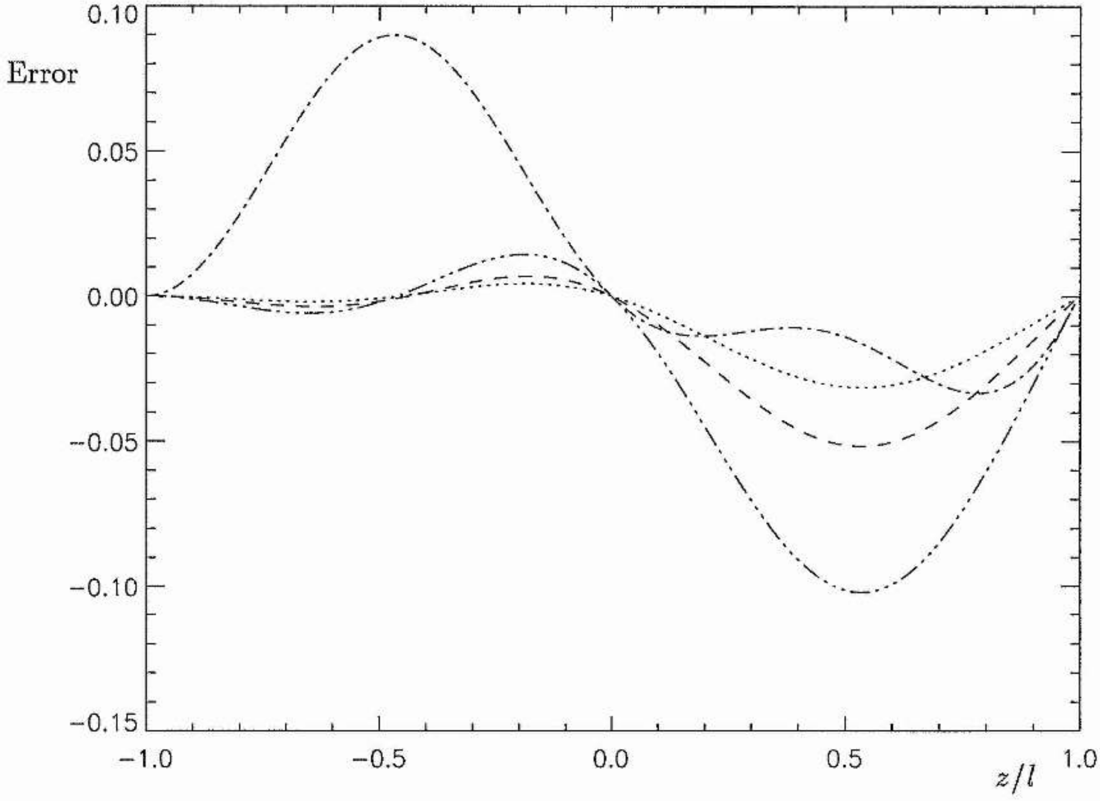

Figure 18: Graph showing the error between the numerical solution and analytical solution for $m = 30, 20, 15$ & 10 with the same line styling as before for the z dependence of the solutions.

As you can see the error is not symmetrical as one would expect. This is because the numerical solution is not an exact match to $A(1 + \cos(\pi z/l))$, but still it is a very good agreement and confirms the form of the solution.

Even though our expression for l_0 is incorrect it does still match that of the expression found for the Gold-Hoyle field in Hood et. al. (1994). We have that

$$l_0 = \frac{2\pi a^2}{c},$$

where a and c are as defined before. Proceeding with the force-free Gold-Hoyle equilibrium we see that

$$l_0 = \frac{\pi}{2}\frac{(1+r^2)^2}{r^2},$$

which evaluated at $r = 1$ gives $l_0 = 2\pi$. Our l is defined as $l = ml_0 + l_1$ and so we have

$$l = 2m\pi + l_1,$$

which is the same as that found in Hood et. al. (1994). It is also seen that the expression found for l_0 for the fully expanded equation also reduces to that of the Gold-Hoyle equilibrium.

6 Conclusions

'Last thing people want is truth'
(Sylvestor Stallone, Antenne 2, 1987)

We have seen that simple analytical expressions can be found to model the critical length of a twisted solar coronal loop but the fact of the matter remains that some of our are expressions, however near, are wrong in some way. We have seen that the numerical data for the force-free Anzer equilibrium exhibits all the elements of the solution we were looking for: the Gaussian shape in the radial direction, the simple cosine function in the z direction, but they also exhibit a quadratic nature (which is expected) but with a constant coefficient of 3/8 and where this arises from is unknown. We can only assume that in finding the analytical solution, something was overlooked. Whether it is in the expansion of the coefficients or in the approximation of the equations to leading order, we have sought, but we have not found.

The results found for the force-free new equilibrium also cannot be explained properly. The evidence suggests that the results are correct, but another argument can be put forward that because such large lengths are being dealt with, the numerical program cannot converge to a finite solution with enough accuracy to say whether or not the solution found is exact. Hence assumptions must be made on the nature of the solutions we have.

However, we have seen a strong agreement in the non-force free results for both the Anzer equilibrium and the new equilibrium, with a confirmation of the results found in Hood et. al. (1994) for the Gold-Hoyle equilibrium.

All in all, it remains that the WKB approximation method is solid in being used to find the critical loop length and further work in this field should be undertaken.

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A Reduction of the marginal stability equations.

We have that,

$$\begin{aligned}
\mathbf{B} &= (0, B_\theta(r), B_z(r)),\\
\boldsymbol{\xi} &= (\xi_r, \frac{-\zeta B_z}{B^2} + \frac{\mu B_\theta}{B^2}, \frac{\zeta B_\theta}{B^2} + \frac{\mu B_z}{B^2}),
\end{aligned}$$

and so we get that

$$\begin{aligned}
\boldsymbol{\xi} \times \mathbf{B} &= (-\zeta, -B_z\xi_r, B_\theta\xi_r),\\
\nabla \times (\boldsymbol{\xi} \times \mathbf{B}) &= \frac{1}{r}(B_\theta \frac{\partial \xi_r}{\partial \theta} + rB_z\frac{\partial \xi_r}{\partial z}, r(-\frac{\partial}{\partial r}(B_\theta\xi_r) - \frac{\partial \zeta}{\partial z}), \frac{\partial}{\partial r}(-rB_z\xi_r) + \frac{\partial \zeta}{\partial \theta}),\\
&= (\mathbf{B}\cdot\nabla\xi_r, -\frac{\partial \zeta}{\partial z} - B_\theta\xi_r' - B_\theta'\xi_r, -B_z\xi_r' - \frac{1}{r}(rB_z)'\xi_r + \frac{1}{r}\frac{\partial \zeta}{\partial \theta})
\end{aligned}$$

and

$$\begin{aligned}
\nabla \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B})) &= [-\frac{B_z}{r}\frac{\partial \xi_r'}{\partial \theta} - \frac{1}{r^2}(rB_z)'\frac{\partial \xi_r}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{\partial^2 \zeta}{\partial z^2} + B_\theta\frac{\partial \xi_r'}{\partial z}\\
&+ B_\theta'\frac{\partial \xi_r}{\partial z},\ \mathbf{B}\cdot\nabla\frac{\partial \xi_r}{\partial z} + B_z\xi_r'' + [B_z' + \frac{1}{r}(rB_z)']\xi_r'\\
&+ (\frac{1}{r}(rB_z)')'\xi_r - \frac{1}{r}\frac{\partial \zeta'}{\partial \theta} + \frac{1}{r^2}\frac{\partial \zeta}{\partial \theta}, -\frac{1}{r}\frac{\partial \zeta}{\partial z} - \frac{B_\theta}{r}\xi_r' - \frac{B_\theta'}{r}\xi_r\\
&- \frac{\partial \zeta'}{\partial z} - 2B_\theta'\xi_r' - B_\theta''\xi_r - B_\theta\xi_r'' - \mathbf{B}\cdot\nabla(\frac{1}{r}\frac{\partial \xi_r}{\partial \theta})].
\end{aligned}$$

Therefore,

$$\begin{aligned}
[\nabla \times (\nabla \times (\boldsymbol{\xi} \times \mathbf{B})) \times \mathbf{B}]_r &= B_z\mathbf{B}\cdot\nabla\frac{\partial \xi_r}{\partial z} + B_z^2\xi_r'' + B_z[B_z' + \frac{1}{r}(rB_z)']\xi_r'\\
&+ B_z(\frac{1}{r}(rB_z)')'\xi_r - \frac{B_z}{r}\frac{\partial \zeta'}{\partial \theta} + \frac{B_z}{r^2}\frac{\partial \zeta}{\partial \theta} + \frac{B_\theta}{r}\frac{\partial \zeta}{\partial z}\\
&+ B_\theta\frac{\partial \zeta'}{\partial z} + B_\theta[\frac{1}{r}(rB_\theta)' + B_\theta']\xi_r' + \frac{B_\theta}{r}(rB_\theta')'\xi_r\\
&+ B_\theta^2\xi_r'' + B_\theta\mathbf{B}\cdot\nabla(\frac{1}{r}\frac{\partial \xi_r}{\partial \theta})
\end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{B}\cdot\nabla)^2\xi_r + B_z^2\xi_r'' + [B_zB_z' + \frac{B_z}{r}(rB_z)' \\
&+ \frac{B_\theta}{r}(rB_\theta)' + B_\theta B_\theta']\xi_r' + B_\theta^2\xi_r'' \\
&+ [B_z(\frac{1}{r}(rB_z'))' + \frac{B_\theta}{r}(rB_\theta')']\xi_r + (B_\theta\frac{\partial}{\partial z} \\
&- \frac{B_z}{r}\frac{\partial}{\partial\theta})\zeta' + \frac{1}{r}(B_\theta\frac{\partial}{\partial z} + \frac{B_z}{r}\frac{\partial}{\partial\theta})\zeta,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
[\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))\times\mathbf{B}]_\theta &= \frac{B_z^2}{r}\frac{\partial\xi_r'}{\partial\theta} + \frac{B_z}{r^2}(rB_z)'\frac{\partial\xi_r}{\partial\theta} - \frac{B_z}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} - B_z\frac{\partial^2\zeta}{\partial z^2} \\
&- B_zB_\theta\frac{\partial\xi_r'}{\partial z} - B_zB_\theta'\frac{\partial\xi_r}{\partial z} \\
&= B_z(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})\xi_r' + [\frac{B_z}{r^2}(rB_z)'\frac{\partial}{\partial\theta} \\
&- B_zB_\theta'\frac{\partial}{\partial z}]\xi_r - \frac{B_z}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} - B_z\frac{\partial^2\zeta}{\partial z^2},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
[\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))\times\mathbf{B}]_z &= B_\theta(B_\theta\frac{\partial}{\partial z} - \frac{B_z}{r}\frac{\partial}{\partial z})\xi_r' + B_\theta[B_\theta'\frac{\partial}{\partial z} \\
&- \frac{1}{r^2}(rB_z)'\frac{\partial}{\partial\theta}]\xi_r + \frac{B_\theta}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} + B_\theta\frac{\partial^2\zeta}{\partial z^2}.
\end{aligned}
\tag{A.3}
$$

Also, we need that

$$
\begin{aligned}
\nabla\times\mathbf{B} &= (0\ ,\ -B_z'\ ,\ \frac{1}{r}(rB_\theta')), \\
\nabla\times\mathbf{B}\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B})) &= [-\frac{B_z'}{r}\frac{\partial\zeta}{\partial\theta} + B_zB_z'\xi_r' + \frac{B_z'}{r}(rB_z)'\xi_r + \frac{1}{r}(rB_\theta)'\frac{\partial\zeta}{\partial z} \\
&+ \frac{B_\theta}{r}(rB_\theta)'\xi_r' + \frac{B_\theta'}{r}(rB_\theta)'\xi_r\ ,\ \frac{1}{r}(rB_\theta)'(\mathbf{B}\cdot\nabla)\xi_r\ , \\
&B_z'(\mathbf{B}\cdot\nabla)\xi_r]
\end{aligned}
$$

$$
\begin{aligned}
&= [(B_zB_z' + \frac{B_\theta}{r}(rB_\theta)')\xi_r' + (\frac{B_z'}{r}(rB_z)' + \frac{B_\theta'}{r}(rB_\theta)')\xi_r \\
&+ (\frac{1}{r}(rB_\theta)'\frac{\partial}{\partial z} - \frac{B_z'}{r}\frac{\partial}{\partial\theta})\zeta\ ,\ \frac{1}{r}(rB_\theta)'(\mathbf{B}\cdot\nabla)\xi_r\ , \\
&\quad (B_z'\mathbf{B}\cdot\nabla)\xi_r]
\end{aligned}
\tag{A.4}
$$

and

$$
\begin{aligned}
\nabla(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi}) &= \nabla(\xi_r p' + \gamma p[\xi_r' + \frac{\xi_r}{r} + \frac{1}{r}(-\frac{B_z}{B^2}\frac{\partial\zeta}{\partial\theta} + \frac{B_\theta}{B^2}\frac{\partial\eta}{\partial\theta}) \\
&+ \frac{B_\theta}{B^2}\frac{\partial\zeta}{\partial z} + \frac{B_z}{B^2}\frac{\partial\zeta}{\partial z}]) \\
&= \nabla(\xi_r p' + \gamma p[\xi_r' + \frac{\xi_r}{r} + \frac{1}{B^2}(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} + B_z\frac{\partial}{\partial z})\eta \\
&- \frac{1}{B^2}(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})\zeta]) \\
&= \nabla(\xi_r p' + \gamma p[\xi_r' + \frac{\xi_r}{r} + \frac{\mathbf{B}\cdot\nabla\eta}{B^2} - \frac{\mathcal{M}\zeta}{B^2}]) \\
&= (\xi_r p' + \gamma p[\xi_r' + \frac{\xi_r}{r} + \frac{\mathbf{B}\cdot\nabla\eta}{B^2} - \frac{\mathcal{M}\zeta}{B^2}])', \\
&\quad \frac{p'}{r}\frac{\partial\xi_r}{\partial\theta} + \frac{\gamma p}{r}\frac{\partial}{\partial\theta}[\xi_r' + \frac{\xi_r}{r} + \frac{\mathbf{B}\cdot\nabla\eta}{B^2} - \frac{\mathcal{M}\zeta}{B^2}], \\
&\quad p'\frac{\partial\xi_r}{\partial z} + \gamma p\frac{\partial}{\partial z}[\xi_r' + \frac{\xi_r}{r} + \frac{\mathbf{B}\cdot\nabla\eta}{B^2} - \frac{\mathcal{M}\zeta}{B^2}]).
\end{aligned}
\tag{A.5}
$$

Then equations $(A.1)$, $(A.2)$, $(A.3)$, $(A.4)$ and $(A.5)$ combine to form the expanded version of equation (3.1). This is a very complicted form of the equation and so needs to be reduced in some way. We do this by forming three separate equations.

Take the scalar product of $\mathbf{B}$ and equation (3.1). The second term gives a zero contribution and so all we are left with is

$$\mathbf{B} \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi}) + \mathbf{B} \cdot \frac{\nabla \times \mathbf{B}}{\mu} \times [\nabla \times (\boldsymbol{\xi} \times \mathbf{B})] = 0.$$

Then by equation $(A.4)$ we get that

$$\mu p' \mathbf{B} \cdot \nabla \xi_r + \mu \gamma p \mathbf{B} \cdot \nabla(\nabla \cdot \boldsymbol{\xi}) + [\frac{B_\theta}{r}(rB_\theta)' + B_z B_z'] \mathbf{B} \cdot \nabla \xi_r = 0. \tag{A.6}$$

From the magnetohydrostatic equation we get that

$$\mu p' + B_z B_z' + \frac{B_\theta}{r}(rB_\theta)' = 0.$$

Then equation $(A.6)$ becomes

$$\mu \gamma p \mathbf{B} \cdot \nabla(\nabla \cdot \boldsymbol{\xi}) = 0, \tag{A.7}$$

which is equation (3.3).

Consider the radial component of (3.1). It is

$$\begin{aligned}
& \mu(p'\xi_r + \gamma p(\nabla \cdot \boldsymbol{\xi}))' + (\mathbf{B} \cdot \nabla)^2 \xi_r + B^2 \xi_r'' + [B_z B_z' + \frac{B_z}{r}(rB_z)' \\
+ \quad & \frac{B_\theta}{r}(rB_\theta)' + B_\theta B_\theta'] \xi_r' \\
+ \quad & [B_z(\frac{1}{r}(rB_z)')' + \frac{B_\theta}{r}(rB_\theta')'] \xi_r - \mathcal{M}\zeta' + \mathcal{N}\zeta + (B_z B_z' + \frac{B_\theta}{r}(rB_\theta)')\xi_r' \\
+ \quad & [\frac{B_z'}{r}(rB_z)' + \frac{B_\theta'}{r}(rB_\theta)'] \xi_r + [\frac{1}{r}(rB_\theta)' \frac{\partial}{\partial z} - \frac{B_z'}{r}\frac{\partial}{\partial \theta}]\zeta \\
= \quad & \mu p' \xi_r' + \mu p'' \xi_r + \gamma p (\nabla \cdot \boldsymbol{\xi})' + \gamma p' (\nabla \cdot \boldsymbol{\xi}) + B^2 \xi_r'' + (\mathbf{B} \cdot \nabla)^2 \xi_r' \\
+ \quad & [-\mu p' + \frac{B^2}{r} + 2B_z B_z' + 2B_\theta B_\theta'] \xi_r' \\
+ \quad & [-\frac{B_z^2}{r^2} + 2\frac{B_z B_z'}{r} + B_z B_z'' + 2\frac{B_\theta B_\theta'}{r} + B_\theta B_\theta'' + B_z'^2 + B_\theta'^2] \xi_r \\
- \quad & \mathcal{M}\zeta' + (\frac{B_z}{r^2}\frac{\partial}{\partial \theta} + 2\frac{B_\theta}{r}\frac{\partial}{\partial z} + B_\theta' \frac{\partial}{\partial z} - \frac{B_z'}{r}\frac{\partial}{\partial \theta})\zeta = 0.
\end{aligned} \tag{A.8}$$

Using the magnetohydrostatic equation, we can gain an expression for p'' thus

$$\mu p'' = \frac{B_\theta^2}{r^2} - B_\theta'^2 - B_\theta B_\theta'' - B_z'^2 - B_z B_z'' - 2\frac{B_\theta B_\theta'}{r}$$

and also remembering that

$$\begin{aligned} \frac{1}{r}(rB^2\xi_r')' &= B^2\xi_r'' + (\frac{B^2}{r} + 2B_zB_z' + 2B_\theta B_\theta')\xi_r', \\ (\mathcal{M}\zeta)' - 2\frac{B_\theta}{r}\frac{\partial\zeta}{\partial z} &= \mathcal{M}\zeta' + \frac{B_z'}{r}\frac{\partial\zeta}{\partial\theta} - \frac{B_z}{r^2}\frac{\partial\zeta}{\partial\theta} - B_\theta'\frac{\partial\zeta}{\partial z} - 2\frac{B_\theta}{r}\frac{\partial\zeta}{\partial z}. \end{aligned}$$

So equation $(A.8)$ becomes

$$\begin{aligned} \frac{1}{r}(rB^2\xi_r')' + \{(\mathbf{B}\cdot\nabla)^2 + \frac{2B_zB_z'}{r} + \frac{B_\theta^2 - B_z^2}{r^2}\}\xi_r = \\ (\mathcal{M}\zeta)' - 2\frac{B_\theta}{r}\frac{\partial\zeta}{\partial z} - (\gamma\mu p\nabla\cdot\xi)', \end{aligned}$$

which is equation (3.4).

For the last equation, take the cross product of the radial component of equation (3.1) with $\mathbf{B}$. We get

$$\begin{aligned} & \nabla(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi})\times\mathbf{B} + (\frac{[\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))]}{\mu}\times\mathbf{B})\times\mathbf{B} \\ + & (\frac{\nabla\times\mathbf{B}}{\mu}\times[\nabla\times(\boldsymbol{\xi}\times\mathbf{B})])\times\mathbf{B} \\ = & 0 \\ & \nabla(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi})\times\mathbf{B} + [\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))\cdot\mathbf{B}]\frac{\mathbf{B}}{\mu} \\ - & (\frac{\mathbf{B}\cdot\mathbf{B}}{\mu})(\nabla\times(\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))) + ((\nabla\times\mathbf{B})\cdot\mathbf{B})\frac{\nabla\times(\boldsymbol{\xi}\times\mathbf{B})}{\mu} \\ - & ((\nabla\times(\boldsymbol{\xi}\times\mathbf{B}))\cdot\mathbf{B})\frac{\nabla\times\mathbf{B}}{\mu} \\ = & 0 \end{aligned}$$

$$
\begin{aligned}
& p'\frac{B_z}{r}\frac{\partial \xi_r}{\partial \theta} + \gamma p \frac{B_z}{r}\frac{\partial}{\partial \theta}(\nabla \cdot \boldsymbol{\xi}) - p' B_\theta \frac{\partial \xi_r}{\partial z} - \gamma p B_\theta \frac{\partial}{\partial z}(\nabla \cdot \boldsymbol{\xi}) \\
- & \frac{B^2}{\mu}\left(-\frac{B_z}{r}\frac{\partial \xi_r'}{\partial \theta} - \frac{1}{r^2}(rB_z)'\frac{\partial \xi_r}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} + \frac{\partial^2 \zeta}{\partial z^2} + B_\theta \frac{\partial \xi_r'}{\partial z} + B_\theta' \frac{\partial \xi_r}{\partial z}\right) \\
+ & \frac{1}{\mu}\left(-B_\theta B_z' + \frac{B_z}{r}(rB_\theta)'\right)\mathbf{B} \cdot \nabla \xi_r \\
= & \ 0 \\
& \mu p'\left(\frac{B_z}{r}\frac{\partial}{\partial \theta} - B_\theta \frac{\partial}{\partial z}\right)\xi_r + \gamma \mu p\left(\frac{B_z}{r}\frac{\partial}{\partial \theta} - B_\theta \frac{\partial}{\partial z}\right)(\nabla \cdot \boldsymbol{\xi}) + B^2\left(\left(\frac{B_z}{r}\frac{\partial}{\partial \theta}\right.\right. \\
- & \left. B_\theta \frac{\partial}{\partial z}\right)\xi_r' + \left(\frac{1}{r^2}(rB_z)'\frac{\partial}{\partial \theta} - B_\theta' \frac{\partial}{\partial z}\right)\xi_r - \mathcal{L}\zeta\Big) + \left(-B_\theta B_z' + \frac{B_z}{r}(rB_\theta)'\right)\mathbf{B} \cdot \nabla \xi_r \\
= & \ 0.
\end{aligned}
$$

Remembering the definitions of $\mathcal{L}$ and $\mathcal{M}$ and expanding the terms in brackets the above reduces to

$$
\begin{aligned}
-B^2\mathcal{L}\zeta + \mu\gamma p\mathcal{M}(\nabla \cdot \boldsymbol{\xi}) + B^2\mathcal{M}\xi_r' + \Big(\frac{B_z^3}{r}\frac{\partial}{\partial \theta} + \frac{B_\theta^3}{r}\frac{\partial}{\partial z} & \\
+ \frac{B_\theta^2 B_z}{r^2}\frac{\partial}{\partial \theta} + \frac{B_z^2 B_\theta}{r}\frac{\partial}{\partial z}\Big)\xi_r = 0. &
\end{aligned}
$$

i.e

$$
-B^2\mathcal{L}\zeta + \mu\gamma p\mathcal{M}(\nabla \cdot \boldsymbol{\xi}) + B^2\mathcal{M}\xi_r' + \frac{B^2}{r}\left(\frac{B_z}{r}\frac{\partial}{\partial \theta} + B_\theta \frac{\partial}{\partial z}\right)\xi_r = 0.
$$

So

$$
\mathcal{L}\zeta = \mathcal{M}\xi_r' + \mathcal{N}\xi_r + \frac{\gamma \mu p}{B^2}\mathcal{M}(\nabla \cdot \boldsymbol{\xi}),
$$

which is equation (3.2).

B *Derivation of the final equation for ξ_r for force-free fields.*

Re-write equation (3.4) as

$$\frac{1}{r}(rB^2\xi_r')' + \mathcal{O}\xi_r = \mathcal{M}\zeta' + \mathcal{P}\zeta - \mu\gamma p(\nabla\cdot\boldsymbol{\xi})' - \mu\gamma p'(\nabla\cdot\boldsymbol{\xi}),$$

where

$$\begin{aligned}
\mathcal{P} &= (\frac{B_z}{r})'\frac{\partial}{\partial\theta} - \frac{1}{r}(r^2B_\theta)'\frac{\partial}{\partial z},\\
\mathcal{O} &= \mathcal{Q}^2 + 2\frac{B_zB_z'}{r} + \frac{B_\theta^2 - B_z^2}{r^2},\\
\mathcal{Q} &= (\mathbf{B}\cdot\nabla).
\end{aligned}$$

Operate on equation (3.2) by $\mathcal{Q}$, giving

$$\mathcal{L}\mathcal{Q}\zeta = \mathcal{M}\mathcal{Q}\xi_r' + \mathcal{N}\mathcal{Q}\xi_r. \tag{B.1}$$

Operate on equation (3.4) by $\mathcal{Q}^2$, to yield

$$\mathcal{Q}^2\frac{1}{r}(rB^2\xi_r')' + \mathcal{O}\mathcal{Q}^2\xi_r = \mathcal{M}\mathcal{Q}^2\zeta' + \mathcal{P}\mathcal{Q}^2\zeta - \gamma\mu p\mathcal{Q}^2(\nabla\cdot\boldsymbol{\xi})',$$

giving

$$\mathcal{Q}^2\frac{1}{r}(rB^2\xi_r')' + \mathcal{O}\mathcal{Q}^2\xi_r = \mathcal{M}\mathcal{Q}^2\zeta' + \mathcal{P}\mathcal{Q}^2\zeta. \tag{B.2}$$

Since $\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi}) = 0$ then we have that

$$\begin{aligned}
&\mathbf{B}\cdot\nabla(\nabla\cdot\boldsymbol{\xi})' + (\mathbf{B}\cdot\nabla)'(\nabla\cdot\boldsymbol{\xi}) = 0,\\
&\Rightarrow\ (\mathbf{B}\cdot\nabla)^2(\nabla\cdot\boldsymbol{\xi})' = 0.
\end{aligned}$$

i.e

$$\mathcal{Q}^2(\nabla\cdot\boldsymbol{\xi})=0.$$

The radial derivative of equation $(B.1)$ is

$$\mathcal{LQ}\zeta'+(\mathcal{LQ})'\zeta=\mathcal{MQ}\xi_r''+[(\mathcal{MQ})'+\mathcal{NQ}]\xi_r'+(\mathcal{NQ})'\xi_r.$$

Operate by $\mathcal{LQ}$ and substitute the form for $\mathcal{LQ}\zeta$ in equation $(B.1)$ giving

$$\begin{aligned}
\mathcal{L}^2\mathcal{Q}^2\zeta'+(\mathcal{LQ})'\mathcal{MQ}\xi_r'+(\mathcal{LQ})'\mathcal{NQ}\xi_r &= \mathcal{LQMQ}\xi_r''+\mathcal{LQ}[(\mathcal{MQ})'+\mathcal{NQ}]\xi_r'\\
&+ \mathcal{LQ}(\mathcal{NQ})'\xi_r\\
\mathcal{L}^2\mathcal{Q}^2\zeta' &= \mathcal{LQMQ}\xi_r''+[\mathcal{LQ}(\mathcal{MQ})'\\
&+ \mathcal{LQNQ}-(\mathcal{LQ})'\mathcal{MQ}]\xi_r'\\
&+ [\mathcal{LQ}(\mathcal{NQ})'-(\mathcal{LQ})'\mathcal{NQ}]\xi_r.
\end{aligned}$$

Operate on equation $(B.2)$ by $\mathcal{L}^2$, giving

$$\begin{aligned}
\mathcal{L}^2\mathcal{Q}^2\frac{1}{r}(rB^2\xi_r')'+\mathcal{L}^2\mathcal{O}\mathcal{Q}^2\xi_r &= \mathcal{M}\mathcal{L}^2\mathcal{Q}^2\zeta'+\mathcal{PLQLQ}\zeta,\\
\mathcal{L}^2\mathcal{Q}^2\frac{1}{r}(rB^2\xi_r')'+\mathcal{L}^2\mathcal{O}\mathcal{Q}^2\xi_r &= \mathcal{M}\{\mathcal{LQMQ}\xi_r''+[\mathcal{LQ}(\mathcal{MQ})'+\mathcal{LNQ}^2\\
&- (\mathcal{LQ})'\mathcal{MQ}]\xi_r'+[\mathcal{LQ}(\mathcal{NQ})'\\
&- (\mathcal{LQ})'\mathcal{NQ}]\xi_r\}+\mathcal{PLQ}[\mathcal{MQ}\xi_r'+\mathcal{NQ}\xi_r].
\end{aligned}$$

Rearranging gives

$$
\begin{aligned}
& \mathcal{Q}^2[\mathcal{L}^2B^2-\mathcal{L}\mathcal{M}^2]\xi_r''+[\mathcal{L}^2\mathcal{Q}^2\frac{1}{r}(rB^2)'-\mathcal{L}\mathcal{Q}\mathcal{M}^2\mathcal{Q}'-\mathcal{L}\mathcal{Q}^2\mathcal{M}\mathcal{M}' \\
- & \mathcal{L}\mathcal{M}\mathcal{N}\mathcal{Q}^2+\mathcal{M}^2\mathcal{L}\mathcal{Q}'\mathcal{Q}+\mathcal{M}^2\mathcal{L}\mathcal{Q}'\mathcal{Q}+\mathcal{M}^2\mathcal{L}'\mathcal{Q}^2-\mathcal{P}\mathcal{L}\mathcal{M}\mathcal{Q}^2]\xi_r'+[-\mathcal{P}\mathcal{L}\mathcal{N}\mathcal{Q}^2 \\
- & \mathcal{M}\mathcal{L}\mathcal{Q}\mathcal{N}\mathcal{Q}'-\mathcal{M}\mathcal{L}\mathcal{Q}^2\mathcal{N}'+\mathcal{M}\mathcal{L}\mathcal{Q}'\mathcal{N}\mathcal{Q}+\mathcal{M}\mathcal{L}'\mathcal{N}\mathcal{Q}^2+\mathcal{L}^2\mathcal{O}\mathcal{Q}^2]\xi_r \\
= & 0.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\mathcal{Q}^2\{[\mathcal{L}^2B^2-\mathcal{L}\mathcal{M}^2]\xi_r''+[\mathcal{L}^2\frac{1}{r}(rB^2)'-\mathcal{L}\mathcal{M}\mathcal{M}'-\mathcal{L}\mathcal{M}\mathcal{N}+\mathcal{M}^2\mathcal{L}'-\mathcal{P}\mathcal{L}\mathcal{M}]\xi_r' \\
+[\mathcal{L}^2\mathcal{O}-\mathcal{P}\mathcal{L}\mathcal{N}-\mathcal{M}\mathcal{L}\mathcal{N}'+\mathcal{M}\mathcal{L}'\mathcal{N}]\xi_r\}=0.
\end{aligned}
$$

Now consider the coefficients of the ξ_r'', ξ_r' and ξ_r terms. Firstly the coefficient of ξ_r'',

$$
\begin{aligned}
\mathcal{L}(\mathcal{L}B^2-\mathcal{M}^2) &= \mathcal{L}(\frac{B_\theta^2}{r^2}\frac{\partial^2}{\partial\theta^2}+B_\theta^2\frac{\partial^2}{\partial z^2}+\frac{B_z^2}{r^2}\frac{\partial^2}{\partial\theta^2}+B_z^2\frac{\partial^2}{\partial z^2}-\frac{B_z^2}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&+ 2\frac{B_\theta B_z}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial z}-B_\theta^2\frac{\partial^2}{\partial z^2}) \\
&= \mathcal{L}\mathcal{Q}^2.
\end{aligned}
\tag{B.3}
$$

Next consider the coefficient of ξ_r'. It is

$$
\mathcal{L}^2\frac{1}{r}(rB^2)'-\mathcal{L}\mathcal{M}\mathcal{M}'-\mathcal{L}\mathcal{M}\mathcal{N}+\mathcal{M}^2\mathcal{L}'-\mathcal{P}\mathcal{L}\mathcal{M}.
$$

Using the fact that

$$
\frac{\mathcal{L}}{r}(r(\mathcal{L}B^2-\mathcal{M}^2))'=\frac{\mathcal{L}}{r}(\mathcal{L}(rB^2)'+\mathcal{L}'rB^2-\mathcal{M}^2-2r\mathcal{M}\mathcal{M}'),
$$

it becomes

$$
\begin{aligned}
&\mathcal{L}^2\frac{1}{r}(rB^2)' - \mathcal{L}\mathcal{M}\mathcal{M}' - \mathcal{L}\mathcal{M}\mathcal{N} + \mathcal{M}^2\mathcal{L}' - \mathcal{P}\mathcal{L}\mathcal{M} \\
= \;& \frac{\mathcal{L}}{r}(r(\mathcal{L}B^2 - \mathcal{M}^2))' + \mathcal{L}\mathcal{M}\mathcal{M}' - \mathcal{L}\mathcal{M}\mathcal{N} - \mathcal{L}'(\mathcal{L}B^2 - \mathcal{M}^2) + \frac{\mathcal{L}\mathcal{M}^2}{r} - \mathcal{P}\mathcal{L}\mathcal{M} \\
= \;& \frac{\mathcal{L}}{r}(r\mathcal{Q}^2)' - \mathcal{L}'\mathcal{Q}^2 - \frac{\mathcal{L}}{r}\mathcal{Q}^2 + \frac{\mathcal{L}^2B^2}{r} + \mathcal{L}\mathcal{M}(\mathcal{M}' - \mathcal{N} - \mathcal{P}),
\end{aligned}
$$

but

$$
\begin{aligned}
\mathcal{M}(\mathcal{M}' - \mathcal{N} - \mathcal{P}) &= \left(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z}\right)\left(\frac{B_\theta}{r}\frac{\partial}{\partial z} - \frac{B_z}{r^2}\frac{\partial}{\partial\theta}\right) \\
&= -\frac{1}{r}\mathcal{M}^2.
\end{aligned}
$$

So the coefficient of ξ_r' is

$$
\begin{aligned}
&\frac{\mathcal{L}}{r}(r\mathcal{Q}^2)' - \mathcal{L}'\mathcal{Q}^2 - \frac{\mathcal{L}}{r}\mathcal{Q}^2 + \frac{\mathcal{L}}{r}(\mathcal{L}B^2 - \mathcal{M}^2) \\
&\qquad = \frac{\mathcal{L}}{r}(r\mathcal{Q}^2)' - \mathcal{L}'\mathcal{Q}^2. \qquad \text{(B.4)}
\end{aligned}
$$

Lastly we look at the coefficient of ξ_r. It is

$$
\begin{aligned}
- \;& \mathcal{L}(\mathcal{M}\mathcal{N}' + \mathcal{P}\mathcal{N}) + \mathcal{L}'\mathcal{M}\mathcal{N} + \mathcal{L}^2\mathcal{O} \\
= \;& -\mathcal{L}[(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})(\frac{B_z'}{r^2}\frac{\partial}{\partial\theta} - 2\frac{B_z}{r^3}\frac{\partial}{\partial\theta} + \frac{B_\theta'}{r}\frac{\partial}{\partial z} - \frac{B_\theta}{r^2}\frac{\partial}{\partial z}) \\
+ \;& (\frac{B_z'}{r}\frac{\partial}{\partial\theta} - \frac{B_z}{r^2}\frac{\partial}{\partial\theta} - 2\frac{B_\theta}{r}\frac{\partial}{\partial z} - B_\theta'\frac{\partial}{\partial z})(\frac{B_z}{r^2}\frac{\partial}{\partial\theta} + \frac{B_\theta}{r}\frac{\partial}{\partial z})] + \mathcal{L}^2\mathcal{O} \\
- \;& \frac{2}{r^3}\frac{\partial^2}{\partial\theta^2}(\frac{B_z^2}{r^3}\frac{\partial^2}{\partial\theta^2} - \frac{B_\theta^2}{r}\frac{\partial^2}{\partial z^2}) \\
= \;& -\mathcal{L}[2\frac{B_zB_z'}{r^3}\frac{\partial^2}{\partial\theta^2} - 3\frac{B_z^2}{r^4}\frac{\partial^2}{\partial\theta^2} - 2\frac{B_zB_\theta}{r^3}\frac{\partial}{\partial\theta}\frac{\partial}{\partial z} - 2\frac{B_\theta B_\theta'}{r}\frac{\partial^2}{\partial z^2} - \frac{B_\theta^2}{r^2}\frac{\partial^2}{\partial z^2}] \\
- \;& 2\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4} + 2\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} + \mathcal{L}^2\mathcal{O}
\end{aligned}
$$

(B.5)

$$
\begin{aligned}
&= -\mathcal{L}[-\frac{\mathcal{Q}^2}{r^2}+\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}+\frac{B_z^2}{r^2}\frac{\partial^2}{\partial z^2}+2\frac{B_zB_z'}{r^3}\frac{\partial^2}{\partial\theta^2}-3\frac{B_z^2}{r^4}\frac{\partial^2}{\partial\theta^2} \\
&- 2\frac{B_\theta B_\theta'}{r}\frac{\partial^2}{\partial z^2}-\frac{B_\theta^2}{r^2}\frac{\partial^2}{\partial z^2}]-2\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4}+2\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}+\mathcal{L}^2\mathcal{O} \\
&= \mathcal{L}^2\mathcal{O}+\frac{\mathcal{L}\mathcal{Q}^2}{r^2}-2\frac{B_z^2}{r^4}\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial\theta^2}-\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}+\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4}+\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}+3\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} \\
&- \mathcal{L}[2\frac{B_zB_z'}{r^3}\frac{\partial^2}{\partial\theta^2}+2\frac{B_\theta B_\theta'}{r^3}\frac{\partial^2}{\partial\theta^2}-2\mathcal{L}\frac{B_\theta B_\theta'}{r}+\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}],
\end{aligned}
\tag{B.6}
$$

but

$$
\begin{aligned}
\mathcal{L}^2\mathcal{O} &= \mathcal{L}^2\mathcal{Q}^2+\mathcal{L}^2\frac{B_zB_z'}{r}+\mathcal{L}^2(\frac{B_\theta^2-B_z^2}{r^2}) \\
&= \mathcal{L}^2\mathcal{Q}^2+2\mathcal{L}^2\frac{B_zB_z'}{r}+\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}+2\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} \\
&+ \frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}-\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4}-2\frac{B_z^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}-\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}
\end{aligned}
$$

and so equation $(B.6)$ becomes

$$
\begin{aligned}
& \mathcal{L}^2\mathcal{Q}^2+2\mathcal{L}^2\frac{B_zB_z'}{r}+\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}+2\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}+\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}-\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4} \\
&- 2\frac{B_z^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}-\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}+\frac{\mathcal{L}\mathcal{Q}^2}{r^2}+2\frac{B_z^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}-\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}+\frac{B_z^2}{r^6}\frac{\partial^4}{\partial\theta^4} \\
&+ \frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}+3\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}-\mathcal{L}[-\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}-2\mathcal{L}\frac{B_\theta B_\theta'}{r}] \\
&= \mathcal{L}^2\mathcal{Q}^2+\frac{\mathcal{L}\mathcal{Q}^2}{r^2}+\frac{\mathcal{L}^2}{r}[2B_zB_z'+B_\theta B_\theta']+\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}+5\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} \\
&+ 2\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}-2\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}+\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}+\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} \\
&= \mathcal{L}^2\mathcal{Q}^2+\frac{\mathcal{L}\mathcal{Q}^2}{r^2}-2\mathcal{L}^2\frac{B_\theta^2}{r^2}+2\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}+6\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}+2\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}-2\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4} \\
&= \mathcal{L}^2\mathcal{Q}^2+\frac{\mathcal{L}\mathcal{Q}^2}{r^2}-2\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4}-4\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}-2\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}+2\frac{B_\theta^2}{r^6}\frac{\partial^4}{\partial\theta^4} \\
&+ 6\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2}+2\frac{B_\theta^2}{r^2}\frac{\partial^4}{\partial z^4}-2\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{L}^2 Q^2 + \frac{\mathcal{L}Q^2}{r^2} + 2\frac{B_\theta^2}{r^4}\frac{\partial^2}{\partial\theta^2}\frac{\partial^2}{\partial z^2} - 2\frac{B_z^2}{r^2}\frac{\partial^4}{\partial z^4} \\
&= \mathcal{L}^2 Q^2 + \frac{\mathcal{L}Q^2}{r^2} - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}[(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})].
\end{aligned} \tag{B.7}
$$

Remembering that $Q^2 = (\mathbf{B}\cdot\nabla)$ then equations $(B.3)$, $(B.4)$ and $(B.7)$ combine to give

$$
\begin{aligned}
&\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r' \\
&+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2 - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}]\xi_r = 0,
\end{aligned} \tag{B.8}
$$

the final equation for ξ_r.

In the case of non force-free fields the final equation for ξ_r can be derived in much the same manner as that for the force-free case.

C *Derivation of the force-free leading order equation*

The equation for ξ_r being dealt with is

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$
$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2 - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}]\xi_r = 0. \quad \text{(C.1)}$$

When dealing to leading order as we are here, some terms in this equation can be neglected, they are

$$\mathcal{L}'(\mathbf{B}\cdot\nabla)^2\xi_r',$$
$$\frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2\xi_r$$

and noting that

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + \frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)'\xi_r' = \frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B}\cdot\nabla)^2\xi_r'],$$

we have that

$$\frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B}\cdot\nabla)^2\xi_r'] + [\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 - \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}]\xi_r = 0.$$

Now assume that ξ_r takes the form

$$\xi_r = \xi_r(r,z)\exp[im(S(r)+\theta-\Phi z)], \quad \text{(C.2)}$$

then each operator acts in the following way

$$\begin{aligned}\frac{\partial^2\xi_r}{\partial z^2} &= -m^2\Phi^2 - 2im\Phi\frac{\partial\xi_r}{\partial z},\\ (\mathbf{B}\cdot\nabla)\xi_r &= im\frac{B_\theta}{r} + B_z(-im\Phi + \frac{\partial\xi_r}{\partial z})\end{aligned}$$

$$
\begin{aligned}
&= B_z \frac{\partial \xi_r}{\partial z}, \\
(\mathbf{B} \cdot \nabla)^2 \xi_r &= B_z^2 \frac{\partial^2 \xi_r}{\partial z^2}, \\
\mathcal{L}\xi_r &= -\frac{m^2}{r^2} - m^2 \Phi^2 \\
&= -\frac{m^2 B^2}{r^2}, \\
\mathcal{L}^2 \xi_r &= \frac{m^4 B^4}{r^4},
\end{aligned}
$$

where each is expressed only in their leading order contributions. Therefore we now have that

$$
\begin{aligned}
\frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B} \cdot \nabla)^2 \xi_r'] &= \frac{-m^2 B^2}{r^3 B_z^2}\frac{\partial}{\partial r}[imrB_z^2 \frac{\partial^2}{\partial z^2}(S' - \Phi' z)\xi_r] \\
&= \frac{-im^3 B^2}{r^2}\frac{\partial}{\partial r}[-2\Phi' \frac{\partial \xi_r}{\partial z} \\
&+ (S' - \Phi' z)\frac{\partial^2 \xi_r}{\partial z^2}] \\
&= \frac{m^4 B^2}{r^2}[-2\Phi'(S' - \Phi' z)\frac{\partial \xi_r}{\partial z} \\
&+ (S' - \Phi' z)^2 \frac{\partial^2 \xi_r}{\partial z^2}] \\
&= \frac{m^4 B^2}{r^2}\frac{\partial}{\partial z}[(S' - \Phi' z)^2 \frac{\partial \xi_r}{\partial z}], \qquad \text{(C.3)}
\end{aligned}
$$

$$
\mathcal{L}^2 (\mathbf{B} \cdot \nabla)^2 \xi_r = \frac{m^4 B^4}{r^4 B_z^2}\frac{\partial^2 \xi_r}{\partial z^2}, \qquad \text{(C.4)}
$$

$$
-\frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B} \cdot \nabla)(B_z \frac{\partial}{\partial z} - \frac{B_\theta}{r}\frac{\partial}{\partial \theta})\}\xi_r = -\frac{4im^3 B_\theta^3}{r^5 B_z}\frac{\partial \xi_r}{\partial z}. \qquad \text{(C.5)}
$$

Then setting $a^2 = B^2/(r^2B_z^2)$ and $c = 4B_\theta^3/(r^3B_zB^2)$, equations $(C.3)$, $(C.4)$ and $(C.5)$ combine to give

$$\frac{\partial}{\partial z}\{[a^2 + (S' - \Phi' z)^2]\frac{\partial \xi_r}{\partial z} - \frac{ic}{m}\xi_r\} = 0, \tag{C.6}$$

the required equation.

Integrating this twice (using an integrating factor) yields

$$\xi_r = A + B\exp\left[\frac{ic}{m\Phi' a}\arctan\left(\frac{S' - \Phi' z}{a}\right)\right].$$

The imposed boundary conditions are $\xi_r = 0$ on $z = \pm l/2$. Thus

$$\begin{aligned} A + B\exp\left[\frac{ic}{m\Phi' a}\arctan\left(\frac{S' - \Phi' l/2}{a}\right)\right] &= 0, \\ A + B\exp\left[\frac{ic}{m\Phi' a}\arctan\left(\frac{S' + \Phi' l/2}{a}\right)\right] &= 0, \end{aligned}$$

which implies that

$$\frac{c}{m\Phi' a}\arctan\left(\frac{S' + \Phi' l/2}{a}\right) = \frac{c}{m\Phi' a}\arctan\left(\frac{S' - \Phi' l/2}{a}\right) \pm 2\pi.$$

We now set

$$\tan\phi = \frac{S' + \Phi' l/2}{a}\ ,\ \tan\theta = \frac{S' - \Phi' l/2}{a},$$

such that $\phi - \theta = \pm m\Phi' a 2\pi/c$.

Therefore

$$\begin{aligned} \tan(\phi - \theta) &= \tan\left(m\Phi' a\frac{2\pi}{c}\right) \\ &= \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta} \\ &= \frac{\frac{S' + \Phi' l/2}{a} - \frac{S' - \Phi' l/2}{a}}{1 + \frac{S'^2 - \Phi'^2 l^2/4}{a^2}}. \end{aligned}$$

So,

$$
\begin{aligned}
\tan{(m\Phi' a\frac{2\pi}{c})} &= \frac{a\Phi' l}{a^2 + S'^2 - \Phi'^2 l^2/4} \\
\Rightarrow \quad S'^2 &= \Phi'^2\frac{l^2}{4} - a^2 + \frac{a\Phi' l}{\tan{(m\Phi' a 2\pi/c)}}.
\end{aligned}
\tag{C.7}
$$

Evaluation of the Bohr - Sommerfeld condition

We also show here the mathematics of evaluating the *Bohr* $-$ *Sommerfeld* condition to find the value of l_1. Equation (4.15) is

$$
\begin{aligned}
S'^2 &= \frac{l_1 c_0}{2\pi m} - \frac{a_0^4}{3c_0^2}\pi^2\Phi_0''^2 x^2 \\
&= A^2 - B^2 x^2
\end{aligned}
\tag{C.8}
$$

say, where $A^2 = l_1 c_0/2\pi m$ and $B^2 = \Phi_0''^2\pi^2 4_0/3c_0^2$. Then, the *Bohr* $-$ *Sommerfeld* condition is

$$m\int_{r_1}^{r_2} S' dr = m\int \frac{dS}{dx}\frac{dx}{dr}dr = m\int_{x_1}^{x_2} m\frac{dS}{dx}\frac{dx}{m} = \frac{\pi}{2}$$

since $dr = dx/m$. Therefore

$$m\int_{x_1}^{x_2}(A^2 - B^2x^2)^{1/2}dx = \frac{\pi}{2}$$

and this is solved easily by putting $x = \frac{A}{B}\sin\theta$. It yields

$$\frac{mA^2}{B} = 1,$$

thus giving

$$l_1 = \frac{2}{\sqrt{3}}\frac{\pi^2 a_0^2 \Phi_0''}{c_0^2}. \tag{C.9}$$

Expansion of the full equation

We now only neglect terms of $O(m^2)$ and less. So,

$$\begin{aligned}
\frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B}\cdot\nabla)^2\xi_r'] &= \frac{-m^2B^2}{r^3B_z^2}\frac{\partial}{\partial r}[rB_z^2(im[-2\Phi'\frac{\partial\xi_r}{\partial z}+(S'-\Phi'z)\frac{\partial^2\xi_r}{\partial z^2}])] \\
&= \frac{-m^2B^2}{r^3B_z^2}[im(B_z^2+rB_zB_z')(-2\Phi'\frac{\partial\xi_r}{\partial z}+(S'-\Phi'z)\frac{\partial^2\xi_r}{\partial z^2}) \\
&+ rB_z^2im\{-2\Phi''\frac{\partial\xi_r}{\partial z}-2\Phi'(\frac{\partial\xi_r'}{\partial z}+im(S'-\Phi'z)\frac{\partial\xi_r}{\partial z}) \\
&+ (S''-\Phi''z)\frac{\partial^2\xi_r}{\partial z^2}+(S'-\Phi'z)(\frac{\partial^2\xi_r'}{\partial z^2}+im(S'-\Phi'z)\frac{\partial^2\xi_r}{\partial z^2})\}] \\
&= \frac{-m^2B^2}{r^3B_z^2}[im(B_z^2+rB_zB_z')(-2\Phi'\frac{\partial\xi_r}{\partial z}+(S'-\Phi'z)\frac{\partial^2\xi_r}{\partial z^2}) \\
&+ imrB_z^2(-2\Phi''\frac{\partial\xi_r}{\partial z}+(S''-\Phi''z)\frac{\partial^2\xi_r}{\partial z^2}) \\
&+ rB_z^2m^2(2\Phi'(S'-\Phi'z)\frac{\partial\xi_r}{\partial z}-(S'-\Phi'z)^2\frac{\partial^2\xi_r}{\partial z^2})] \\
&= \frac{m^4B^2}{r^2B_z^2}[\frac{-i}{mr}(B_z^2+rB_zB_z')(\frac{\partial}{\partial z}(S'-\Phi'z)\frac{\partial\xi_r}{\partial z}-\Phi'\frac{\partial\xi_r}{\partial z}) \\
&- \frac{i}{m}B_z^2(\frac{\partial}{\partial z}(S''-\Phi''z)\frac{\partial\xi_r}{\partial z}-\Phi''\frac{\partial\xi_r}{\partial z}) \\
&+ B_z^2(\frac{\partial}{\partial z}(S'-\Phi'z)^2\frac{\partial\xi_r}{\partial z},
\end{aligned}$$

$$\mathcal{L}'(\mathbf{B}\cdot\nabla)^2\xi_r' = \frac{2im^3}{r^3}B_z^2(\frac{\partial}{\partial z}(S'-\Phi'z)\frac{\partial\xi_r}{\partial z}-\Phi'\frac{\partial\xi_r}{\partial z}),$$

$$\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2\xi_r = \frac{m^4B^4}{r^4B_z^2}\frac{\partial^2\xi_r}{\partial z^2},$$

$$\frac{-2}{r^2}\frac{\partial^2}{\partial z^2}\{(\mathbf{B}\cdot\nabla)(B_z\frac{\partial}{\partial z}-\frac{B_\theta}{r}\frac{\partial}{\partial\theta})\}\xi_r = \frac{-4im^3B_\theta^3}{r^5B_z}\frac{\partial\xi_r}{\partial z}.$$

Then equation (4.7) becomes,

$$
\begin{aligned}
& \frac{\partial}{\partial z}\{(\frac{m^4B^4}{r^4B_z^2}+\frac{m^4B^2}{r^2B_z^2}(B_z^2(S'-\Phi'z)^2-\frac{i}{mr}(B_z^2+rB_zB_z')(S'-\Phi'z) \\
- & \frac{i}{m}B_z^2(S''-\Phi''z))-\frac{2im^3B_z^2}{r^3}(S'-\Phi'z))\frac{\partial\xi_r}{\partial z} \\
+ & (\frac{im^3B^2}{r^3B_z^2}(\Phi'(B_z^2+rB_zB_z')+rB_z^2\Phi'')+\frac{2im^3B_z^2}{r^3}\Phi'-\frac{4im^3B_\theta^3}{r^5B_z})\xi_r\} \\
= & \ 0, \\
& \frac{\partial}{\partial z}\{(\frac{B^2}{r^2B_z^2}+(S'-\Phi'z)^2-\frac{i}{mr}(S'-\Phi'z)(1+\frac{rB_zB_z'}{B_z^2}+\frac{2B_z^2}{B^2}) \\
- & \frac{i}{m}(S''-\Phi''z))\frac{\partial\xi_r}{\partial z}+(\frac{i}{mr}\Phi'(1+\frac{rB_zB_z'}{B_z^2}+\frac{2B_z^2}{B^2})+\frac{i}{m}\Phi''-\frac{4iB_\theta^3}{mr^3B_zB^2})\xi_r\} \\
= & \ 0, \\
& \frac{\partial}{\partial z}\{(a^2+(S'-\Phi'z)^2-\frac{i}{m}b(S'-\Phi'z)-\frac{i}{m}(S''-\Phi''z))\frac{\partial\xi_r}{\partial z} \\
+ & \frac{i}{m}(b\Phi'+\Phi''-c)\xi_r\} \\
= & \ 0,
\end{aligned}
$$

where a^2 & c are as previously defined and

$$b=1+\frac{rB_zB_z'}{B_z^2}+\frac{2B_z^2}{B^2}.$$

If we now write

$$
\begin{aligned}
z &= l\bar{z} \\
&= ml_0\bar{z}, \\
m\Phi' &= x\Phi_0'',
\end{aligned}
$$

we get

$$
\begin{aligned}
& \frac{\partial}{\partial \bar{z}}\{(a^2+(S'-\Phi_0''xl_0\bar{z})^2-\frac{i}{m}b(S'-\Phi_0''xl_0\bar{z})-\frac{i}{m}(S''-\Phi_0''ml_0\bar{z}))\frac{1}{ml_0}\frac{\partial \xi_r}{\partial \bar{z}} \\
+ & \frac{i}{m}(\frac{b\Phi_0''x}{m}+\Phi_0''-c)\xi_r\} \\
= & \ 0.
\end{aligned}
$$

The boundary conditions tell us that at $x=0$, $S'=0$ and so we have

$$\frac{\partial}{\partial \bar{z}}\{(a^2-\frac{i}{m}(S'-\Phi_0''ml_0\bar{z}))\frac{1}{ml_0}\frac{\partial \xi_r}{\partial \bar{z}}-\frac{i}{m}(-\Phi_0''+c)\xi_r\}=0.$$

We can assume that $S''=0$ and so integrating twice yields

$$\xi_r = A + B\exp\left[-\frac{(c-\Phi_0'')}{\Phi_0''}\ln\left(a^2+i\Phi_0''l_0\bar{z}\right)\right],$$

where A and B are arbitrary constants. The appropriate boundary conditions are that $\xi_r=0$ at $\bar{z}=\pm 1/2$ and so we have that

$$A + B\exp\left[-\frac{(c-\Phi_0'')}{\Phi_0''}\ln\left(a^2+i\Phi_0''\frac{l_0}{2}\right)\right]=0.$$

Now,

$$\ln\left(a^2+i\Phi_0''\frac{l_0}{2}\right)=ln(\alpha)\ \pm\ i\phi,$$

where

$$\tan\phi \ = \ \frac{\Phi_0''l_0}{2a^2}.$$

We also know that

$$2\phi\frac{(c-\Phi_0'')}{\Phi_0''}=2\pi,$$

giving

$$\phi = \frac{\Phi_0''\pi}{c - \Phi_0''}$$

and so

$$\begin{aligned}
\tan\phi &= \tan\left(\frac{\Phi_0''\pi}{c - \Phi_0''}\right) \\
&= \Phi_0''\frac{l_0}{2a^2} \\
\Rightarrow\ l_0 &= \frac{2a^2}{\Phi_0''}\tan\left(\frac{\Phi_0''\pi}{c - \Phi_0''}\right)
\end{aligned}$$

D *Derivation of the non force-free leading order equation*

The equation that we are dealing with is

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$
$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 - \frac{2p'}{r}\mathcal{L}\frac{\partial^2}{\partial z^2} + \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{\mathbf{B}\cdot\nabla(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} - B_z\frac{\partial}{\partial z})\} + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2]\xi_r$$
$$+2\frac{B_\theta}{rB^4}(\frac{B_z}{r}\frac{\partial}{\partial\theta} - B_\theta\frac{\partial}{\partial z})^3\frac{\partial}{\partial z}(\gamma\mu p\nabla\cdot\boldsymbol{\xi}) = 0.$$

Now, with a trial solution of the form that we are using, i.e

$$\xi_r = \xi_r(r,z)\exp\left[im(S(r) + \theta - \Phi z)\right],$$

we can neglect a few terms of the equation for ξ_r since to leading order they do not contribute. These terms are

$$\mathcal{L}'(\mathbf{B}\cdot\nabla)^2\xi_r',$$
$$\frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{\mathbf{B}\cdot\nabla(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} - B_z\frac{\partial}{\partial z})\}\xi_r,$$
$$\frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2\xi_r.$$

Hence we write the equation as

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + \frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)'\xi_r' + \mathcal{L}^2(\mathbf{B}\cdot\nabla)^2\xi_r - \frac{2p'}{r}\mathcal{L}\frac{\partial^2\xi_r}{\partial z^2}$$
$$+2\frac{B_\theta}{rB^2}\frac{\mathcal{M}^2}{B^2}\mathcal{M}\frac{\partial}{\partial z}(\gamma\mu p\nabla\cdot\boldsymbol{\xi}) = 0, \qquad \text{(D.1)}$$

we also have that

$$(\nabla\cdot\boldsymbol{\xi}) = \frac{\mathbf{B}\cdot\nabla\eta}{B^2} - \frac{\mathcal{M}\zeta}{B^2} + \xi_r' + \frac{\xi_r}{r}.$$

Operating by $\mathcal{M}$ we get

$$\mathcal{M}(\nabla \cdot \boldsymbol{\xi}) = \mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - \frac{\mathcal{M}^2 \zeta}{B^2} + \mathcal{M}\xi_r' + \frac{\mathcal{M}\xi_r}{r}. \tag{D.2}$$

Subtracting equation (3.2) from this yields

$$\mathcal{M}(\nabla \cdot \boldsymbol{\xi}) - \mathcal{L}\zeta = \mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - \frac{\mathcal{M}^2 \zeta}{B^2} + (\frac{\mathcal{M}}{r} - \mathcal{N})\xi_r - \gamma\mu p \frac{\mathcal{M}}{B^2}(\nabla \cdot \boldsymbol{\xi})$$

and rearranging gives

$$\begin{aligned} (1 + \frac{\gamma\mu p}{B^2}\mathcal{M}(\nabla \cdot \boldsymbol{\xi})) &= \mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z} + (\mathcal{L} - \frac{\mathcal{M}^2}{B^2})\zeta \\ &= \mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z} + \frac{(\mathbf{B} \cdot \nabla)^2}{B^2}\zeta \\ &= \mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z} \end{aligned} \tag{D.3}$$

and so by operating by $(\mathbf{B} \cdot \nabla)$ we get

$$0 = (\mathbf{B} \cdot \nabla)\{\mathcal{M}\frac{\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z}\}. \tag{D.4}$$

Using equation $(D.3)$, equation $(D.1)$ becomes

$$\begin{aligned} & \frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B} \cdot \nabla)^2 \xi_r'] + \mathcal{L}^2(\mathbf{B} \cdot \nabla)^2 \xi_r - 2\frac{p'}{r}\mathcal{L}\frac{\partial^2 \xi_r}{\partial z^2} \\ + & \; 2\frac{B_\theta}{r}\frac{\mathcal{M}^2}{B^2}\frac{\gamma\mu p}{B^2 + \gamma\mu p}\frac{\partial}{\partial z}\{\frac{\mathcal{M}\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z}\} \\ = & \; 0 \\ & \frac{\mathcal{L}}{r}\frac{\partial}{\partial r}[r(\mathbf{B} \cdot \nabla)^2 \xi_r'] + \mathcal{L}(\frac{\mathcal{M}^2}{B^2} + \frac{(\mathbf{B} \cdot \nabla)^2}{B^2})(\mathbf{B} \cdot \nabla)^2 \xi_r - 2\frac{p'}{r}\mathcal{L}\frac{\partial^2 \xi_r}{\partial z^2} \\ + & \; 2\frac{B_\theta}{r}(\mathcal{L} - \frac{(\mathbf{B} \cdot \nabla)^2}{B^2})\frac{\gamma\mu p}{B^2 + \gamma\mu p}\frac{\partial}{\partial z}\{\frac{\mathcal{M}\mathbf{B} \cdot \nabla \eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial \xi_r}{\partial z}\} \\ = & \; 0 \end{aligned}$$

$$
\begin{aligned}
& \frac{1}{r}\frac{\partial}{\partial r}[r(\mathbf{B}\cdot\nabla)^2\xi_r'] + \frac{\mathcal{M}^2}{B^2}(\mathbf{B}\cdot\nabla)^2\xi_r - 2\frac{p'}{r}\frac{\partial^2\xi_r}{\partial z^2} \\
+ \quad & 2\frac{B_\theta}{r}\frac{\gamma\mu p}{B^2+\gamma\mu p}\frac{\partial}{\partial z}\{\frac{\mathcal{M}\mathbf{B}\cdot\nabla\eta}{B^2} - 2\frac{B_\theta}{r}\frac{\partial\xi_r}{\partial z}\} \\
= \quad & 0. \qquad\qquad (D.5)
\end{aligned}
$$

Assuming a solution of the form

$$
\xi_r = \xi_r(r,z)\exp im(S(r)+\theta-\Phi z),
$$

the operators transform the equation by

$$
\begin{aligned}
\frac{\partial\xi_r}{\partial z} &= -im\Phi\xi_r + \frac{\partial\xi_r}{\partial z}, \\
\frac{\partial^2\xi_r}{\partial z^2} &= -m^2\Phi^2\xi_r - 2im\Phi\frac{\partial\xi_r}{\partial z}, \\
\mathcal{M}\xi_r &= im\frac{B_z}{r}\xi_r - B_\theta(-im\Phi + \frac{\partial\xi_r}{\partial z}) \\
&= \frac{imB^2}{rB_z}\xi_r - B_\theta\frac{\partial\xi_r}{\partial z},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{M}^2\xi_r}{B^2} &= -\frac{m^2B^2}{r^2B_z}\xi_r - 2\frac{imB_\theta}{rB_z}\frac{\partial\xi_r}{\partial z}, \\
(\mathbf{B}\cdot\nabla)\xi_r &= im\frac{B_\theta}{r}\xi_r + B_z(-im\Phi + \frac{\partial\xi_r}{\partial z}) \\
&= B_z\frac{\partial\xi_r}{\partial z}, \\
(\mathbf{B}\cdot\nabla)^2\xi_r &= B_z^2\frac{\partial^2\xi_r}{\partial z^2},
\end{aligned}
$$

where each is expressed in their leading order terms only.

Equation $(D.5)$ becomes

$$
\begin{aligned}
& \frac{1}{r}\frac{\partial}{\partial r}[rB_z^2\frac{\partial^2}{\partial z^2}(\xi_r' + im(S' - \Phi' z)\xi_r)] + (-\frac{m^2B^2}{r^2B_z^2} - 2\frac{imB_\theta}{rB_z}\frac{\partial}{\partial z})(B_z^2\frac{\partial^2}{\partial z^2})\xi_r \\
+ & \ 2\frac{p'}{r}(-m^2\Phi^2 - 2im\Phi\frac{\partial}{\partial z})\xi_r + 2\frac{B_\theta}{r}\frac{\gamma\mu p}{B^2+\gamma\mu p}(-im\Phi + \frac{\partial}{\partial z})\{(\frac{imB^2}{rB_z} \\
- & \ B_\theta\frac{\partial}{\partial z})\frac{B_z}{B^2}\frac{\partial\eta}{\partial z} \ - 2\frac{B_\theta}{r}(-im\Phi + \frac{\partial}{\partial z})\xi_r\} \\
= & \ 0 \\
& B_z^2\frac{\partial^2\xi_r''}{\partial z^2} - m^2B_z^2(S' - \Phi' z)^2\frac{\partial^2\xi_r}{\partial z^2} - \frac{m^2B^2}{r^2}\frac{\partial^2\xi_r}{\partial z^2} - 2\frac{imB_\theta B_z}{r}\frac{\partial^3\xi_r}{\partial z^3} \\
- & \ 2\frac{p'}{r}\frac{m^2B_\theta^2}{r^2B_z^2}\xi_r - 4im\frac{p'B_\theta}{r^2B_z}\frac{\partial\xi_r}{\partial z} + 2\frac{B_\theta}{r}\frac{\gamma\mu p m^2}{B^2+\gamma\mu p}\{\frac{B_\theta}{r^2B_z}\frac{\partial\eta}{\partial z} + 2\frac{B_\theta^3}{r^3B_z^2}\xi_r\} \\
+ & \ 2\frac{B_\theta}{r}\frac{\gamma\mu p}{B^2+\gamma\mu p}\{(\frac{m^2B^2B_\theta}{r^2B_z^2} + \frac{imB_\theta^2}{rB_z} + \frac{imB^2}{rB_z}\frac{\partial}{\partial z} - B_\theta\frac{\partial^2}{\partial z^2})\frac{B_z}{B^2}\frac{\partial\eta}{\partial z} \\
- & \ 2\frac{B_\theta}{r}(-\frac{m^2B_\theta^2}{r^2B_z^2} - 2\frac{imB_\theta}{rB_z}\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2})\xi_r\} \\
= & \ 0.
\end{aligned}
$$

Writing

$$
\tilde{\eta} = \frac{rB_z}{2B_\theta^2}\eta,
$$

equation $(D.5)$ reduces to

$$
\begin{aligned}
& B_z^2\frac{\partial^2\xi_r''}{\partial z^2} - m^2B_z^2(S' - \Phi' z)^2\frac{\partial^2\xi_r}{\partial z} - \frac{m^2B^2}{r^2}\frac{\partial^2\xi_r}{\partial z^2} - 2\frac{imB_\theta B_z}{r}\frac{\partial^3\xi_r}{\partial z^3} \\
- & \ 2m^2p'\frac{B_\theta^2}{r^3B_z}\xi_r - 4imp'\frac{B_\theta}{r^2B_z}\frac{\partial\xi_r}{\partial z} + 4\frac{B_\theta^4}{r^4B_z^2}\frac{\gamma\mu p m^2}{B^2+\gamma\mu p}\{\frac{\partial\tilde{\eta}}{\partial z} + \xi_r\} \\
+ & \ \frac{4imB_\theta^3}{r^3B_z}\frac{\gamma\mu p}{B^2+\gamma\mu p}\frac{\partial}{\partial z}\{\frac{\partial\tilde{\eta}}{\partial z} + \xi_r\} \\
= & \ 0.
\end{aligned}
$$

The $O(m^2)$ equation is then

$$-B_z^2(S' - \Phi' z)^2 \frac{\partial^2 \xi_r}{\partial z^2} - \frac{B^2}{r^2}\frac{\partial^2 \xi_r}{\partial z^2} - 2p' \frac{B_\theta^2}{r^3 B_z^2}\xi_r + \frac{4B_\theta^4}{r^4 B_z^2}\frac{\gamma\mu p}{B^2 + \gamma\mu p}\{\frac{\partial \tilde{\eta}}{\partial z} + \xi_r\}$$
$$= 0.$$

Then writing

$$a = \frac{B^2}{r^2}\ ,\ b = B_z^2\ ,\ c = 2p'\frac{B_\theta^2}{r^3 B_z^2}\ ,\ d = \frac{4B_\theta^4 \gamma\mu p}{r^4 B_z^2(B^2 + \gamma\mu p)},$$

the resulting equation is

$$\frac{\partial}{\partial z}[(a + b(S' - \Phi' z)^2)\frac{\partial \xi_r}{\partial z}] + c\xi_r - d(\frac{\partial \tilde{\eta}}{\partial z} + \xi_r) = 0, \qquad \text{(D.6)}$$

which is equation (4.25).

E *Mathematical explanation for the non force-free results*

Equation (4.24) can written in the form

$$\mathcal{L}(\mathbf{B}\cdot\nabla)^2\xi_r'' + [\frac{\mathcal{L}}{r}(r(\mathbf{B}\cdot\nabla)^2)' - \mathcal{L}'(\mathbf{B}\cdot\nabla)^2]\xi_r'$$
$$+[\mathcal{L}^2(\mathbf{B}\cdot\nabla)^2 - \frac{2p'}{r}\mathcal{L}\frac{\partial^2}{\partial z^2} + \frac{2}{r^2}\frac{\partial^2}{\partial z^2}\{\mathbf{B}\cdot\nabla(\frac{B_\theta}{r}\frac{\partial}{\partial\theta} - B_z\frac{\partial}{\partial z})\} + \frac{\mathcal{L}}{r^2}(\mathbf{B}\cdot\nabla)^2]\xi_r$$
$$+2\frac{B_\theta}{r}\frac{\mathcal{M}^2}{B^2}\frac{\partial}{\partial z}\frac{\gamma\mu p}{(B^2+\gamma\mu p)}(\mathcal{M}\mathbf{B}\cdot\nabla\eta - 2\frac{B_\theta}{r}\frac{\partial}{\partial z}\xi_r) = 0.$$

We now assume a solution of the form

$$\xi_r = f(x)F(z, r_0, z_0)\exp[im(\theta - \Phi(r)(z - z_0)], \tag{E.1}$$

where $x = m^{1/2}(r - r_0)$. Following the method of Connor, Hastie and Taylor (1979), the equations can be written in the form

$$A\xi_r + dE = 0, \tag{E.2}$$

where

$$E = \frac{\partial\tilde{\eta}}{\partial z} + \xi_r$$

with

$$\tilde{\eta} = \frac{rB_z}{2B_\theta^2 l}\eta \quad , \quad \frac{\partial}{\partial z}(E) = 0$$

and A is an operator of the form

$$A_0(\frac{\partial}{\partial z}, z; r, z_0) + \frac{1}{m^{1/2}}A_1(\frac{\partial}{\partial z}, z; r, z_0) + \frac{1}{m}A_2(\frac{\partial}{\partial z}, z; r, z_0).$$

We now set ξ_r and η such that

$$\begin{aligned}\xi_r &= \xi_0 + \frac{1}{m^{1/2}}\xi_1 + \frac{1}{m}\xi_2,\\ \eta &= \eta_0 + \frac{1}{m^{1/2}}\eta_1 + \frac{1}{m}\eta_2,\end{aligned}$$

so that on expansion, the order m^4, $m^{7/2}$ and m^3 equations are

$$A_0\xi_0 + d_0E_0 = 0, \tag{E.3}$$

$$A_0\xi_1 + d_0E_1 + A_1\xi_0 = 0, \tag{E.4}$$

$$A_0\xi_2 + d_0E_2 + A_1\xi_1 + A_2\xi_0 + d_2E_0 = 0, \tag{E.5}$$

where

$$\begin{aligned}
A_0 &= B_z^2\Phi'^2[\frac{\partial}{\partial\bar{z}}(\bar{z}-z_0)\frac{\partial}{\partial\bar{z}}] + \frac{B^2}{r^2l^2}\frac{\partial^2}{\partial\bar{z}^2} - \frac{2p'}{r}\Phi^2,\\
A_1 &= \hat{A}_1 i\frac{\partial}{\partial x}\ ,\ \hat{A}_1 = -\frac{1}{\Phi'}\frac{\partial A_0}{\partial z_0},\\
A_2 &= \tilde{A}_2 + \hat{A}_2\frac{\partial^2}{\partial x^2}\ ,\ \hat{A}_2 = \frac{1}{2\Phi'}\frac{\partial}{\partial z_0}(\frac{1}{\Phi'}\frac{\partial A_0}{\partial z_0}),\\
\tilde{A}_2 &= [\frac{1}{r}(rB_z^2)'i\Phi' + B_z^2 i\Phi'' + 2i\frac{\Phi' B_z^4}{rB^2}]\{\frac{\partial}{\partial\bar{z}}(\bar{z}-z_0)\frac{\partial}{\partial\bar{z}}\}\\
&+ [\frac{1}{r}(rB_z^2)'i\Phi' + B_z^2 i\Phi'' + 2i\frac{\Phi' B_z^4}{rB^2} + 4ip'\frac{\Phi}{r}]\frac{\partial}{\partial\bar{z}},\\
d_0 &= 4B_\theta^2\frac{\Phi^2}{r^2}\frac{\gamma\mu p}{(B^2+\gamma\mu p)},\\
d_2 &= 4iB_\theta^2\frac{\Phi}{r^2}\frac{\gamma\mu p}{(B^2+\gamma\mu p)}\frac{\partial}{\partial\bar{z}},\\
z &= l\bar{z}.
\end{aligned}$$

The operators are written in the same form as those in Connor, Hastie and Taylor (1979). Differentiating equation (E.3) with respect to z_0 gives

$$\frac{\partial A_0}{\partial z_0}\xi_0 + A_0\frac{\partial\xi_0}{\partial z_0} + \frac{\partial A_0}{\partial l_0}\frac{\partial l_0}{\partial z_0}\xi_0 + d_0\frac{\partial E_0}{\partial z_0} = 0.$$

Multiplying this by ξ_0 and forming the inner products yields

$$\frac{\partial l_0}{\partial z_0} = 0, \tag{E.6}$$

a solvability condition. Equation (E.4) can be written as

$$A_0\xi_1 + d_0 E_1 = -\frac{\partial}{\partial x}(A_0\frac{\partial \xi_0}{\partial z_0} + d_0\frac{\partial E_0}{\partial z_0}),$$

which gives us that

$$\xi_1 = -\frac{\partial}{\partial x}\frac{\partial \xi_0}{\partial z_0}. \tag{E.7}$$

We now differentiate equation (E.3) twice with respect to z_0 and apply the condition (E.6) to give

$$\frac{\partial^2 A_0}{\partial z_0^2}\xi_0 + \frac{\partial A_0}{\partial l_0}\frac{\partial^2 l_0}{\partial z_0^2}\xi_0 + 2\frac{\partial A_0}{\partial z_0}\frac{\partial \xi_0}{\partial z_0} + A_0\frac{\partial^2 \xi_0}{\partial z_0^2} + d_0\frac{\partial^2 E_0}{\partial z_0^2} = 0. \tag{E.8}$$

For simplicity we now take the exponential part of ξ_0 as written and proceed by multiplying the above by ξ_0 and form the inner products, giving

$$< F_0\frac{\partial A_0}{\partial z_0}\frac{\partial F_0}{\partial z_0} >= -\frac{1}{2} < F_0(\frac{\partial^2 A_0}{\partial z_0^2} - \frac{\partial A_0}{\partial l_0}\frac{\partial^2 l_0}{\partial z_0^2})F_0 > . \tag{E.9}$$

We can also write the operator A_0 in the form

$$A_0 = [\frac{a^2}{l_0^2} + ma^2(\frac{1}{l^2} - \frac{1}{l_0^2})]\frac{\partial^2}{\partial \bar{z}^2} + \Phi'^2\frac{\partial}{\partial \bar{z}}(\bar{z} - z_0)^2\frac{\partial}{\partial \bar{z}} - 2\Phi^2\frac{p'}{r},$$

so that equation (E.5) takes the form

$$A_0\xi_2 + d_0 E_2 = -\frac{\partial A_0}{\partial z_0}f''\frac{\partial F_0}{\partial z_0} - \hat{A}_2 f'' F_0 - \tilde{A}_2 f F_0 - d_2 E_0 - a^2(\frac{1}{l^2} - \frac{1}{l_0^2})\frac{\partial^2}{\partial \bar{z}^2} f F_0. \tag{E.10}$$

Multiplying by ξ_0, forming the inner products and adding $< d_0\eta_0(\partial/\partial\bar{z})E_2 >$ leaves us with

$$< F_0\frac{\partial A_0}{\partial z_0}\frac{\partial F_0}{\partial z_0} > f'' + < F_0\hat{A}_2F_0 > f'' + a^2(\frac{1}{l^2} - \frac{1}{l_0^2}) < F_0\frac{\partial^2 F_0}{\partial \bar{z}^2} > f$$
$$+ < F_0\tilde{A}_2F_0 > f + < F_0 d_2 E_0 > = 0, \quad \text{(E.11)}$$

since

$$\begin{aligned}
& < \xi_0 A_0 \xi_2 + \xi_0 d_0 E_2 > + < d_0\eta_0\frac{\partial}{\partial\bar{z}}E_2 > \\
= \;& < \xi_2 A_0 \xi_0 + \xi_0 d_0(\frac{\partial\eta_2}{\partial\bar{z}} + \xi_2 + \gamma\frac{\partial\xi_0}{\partial\bar{z}}) + d_0\frac{\partial\eta_0}{\partial\bar{z}}(\frac{\partial\eta_2}{\partial\bar{z}} + \xi_2 + \gamma\frac{\partial\xi_0}{\partial\bar{z}}) > \\
= \;& < \xi_2(A_0\xi_0 + d_0E_0) + d_0\frac{\partial\eta_2}{\partial\bar{z}}(\frac{\partial\eta_0}{\partial\bar{z}} + \xi_0) + \gamma d_0\frac{\partial\xi_0}{\partial\bar{z}}(\frac{\partial\eta_0}{\partial\bar{z}} + \xi_0) > \\
= \;& 0, \qquad \text{(E.12)}
\end{aligned}$$

where

$$\gamma = -\frac{irB_z^3}{l_0B_\theta B^2}$$

It is also seen that $\tilde{A}_2$ forms an odd function and so upon integration will become zero. Thus, using the fact that $d_2E_0 = 0$ and (E.9), equation (E.11) becomes

$$\frac{1}{2} < F_0(\frac{\partial^2 A_0}{\partial z_0^2} - \frac{\partial A_0}{\partial l_0}\frac{\partial^2 l_0}{\partial z_0^2})F_0 > f'' - < F_0\frac{\partial^2 F_0}{\partial\bar{z}^2} > f''$$
$$-a^2(\frac{1}{l^2} - \frac{1}{l_0^2}) < F_0\frac{\partial^2 F_0}{\partial\bar{z}^2} > f = 0.$$

Setting $l = l_{min} + l_1/m$ and $l_0 = l_{min} + (\partial^2 l_0/\partial r_0^2)(r - r_0)^2/2$, where $l_{min} = l(r_0, z_0)$, we can re-write the above as

$$\frac{\partial^2 l_0}{\partial z_0^2}\frac{\partial^2 f}{\partial x^2} + \frac{\alpha}{\beta}(2l_1 - \frac{\partial^2 l_0}{\partial r_0^2}x^2)f = 0, \qquad \text{(E.13)}$$

where

$$
\begin{aligned}
\alpha &= \frac{a^2}{l_{min}^3} < F_0 \frac{\partial^2 F_0}{\partial z_0^2} >, \\
\beta &= \frac{1}{2} < F_0 [\frac{(\partial^2 A_0 / \partial z_0^2)}{(\partial^2 l_0 / \partial z_0^2)} - \frac{\partial A_0}{\partial l_0}] F_0 > - \frac{< F_0 (\partial^2 F_0 / \partial \bar{z}^2) >}{(\partial^2 l_0^2 / \partial z_0^2)}.
\end{aligned}
\tag{E.14}
$$

Equation (E.13) is of the same form as that found in Connor, Hastie and Taylor (1979) and is the reason that the form of equation (4.25) should be $(S'^2 + \Phi'^2 z^2)$ and not $(S' - \Phi' z)^2$.

Whenever life gets you down, Mrs Brown, and things seem
hard or tough, and people are stupid obnoxious or daft,
and you feel that you've had quite enough...

Just remember that you're standing on a planet that's evolving,
And revolving at 900 miles an hour,
That's orbiting at 19 miles a second, so it's reckoned,
A sun that is the source of all our power.
The sun and you and me and all the stars that we can see,
Are moving at a million miles a day
In an outer spiral arm at 40,000 miles an hour,
Of the galaxy we call the Milky Way.

Our galaxy itself contains 100 billion stars
It's 100,000 light years side to side,
It bulges in the middle, 16,000 light years thick,
But out by us it's just 3,000 light years wide
We're 30,000 light years from galactic central point,
We go round every 200 million years
And our galaxy is only one of millions and billions
In this amazing and expanding universe.

(Monty Python's Galaxy Song. Composed by Eric Idle & John Du Prez.)